PROJECTIVE GEOMETRY

ANGELO ANDES ROVIDA

EDITED BY NORMAN DAVIDSON

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INTRODUCTION to PROJECTIVE GEOMETRY

working notes for students

No use is made here of formalization or of algebra. It is all the more important that the reader should execute the drawings himself. Processes, movements and transformations have to be done in one's imagination. The best way to help this to happen is to draw with one's own hand in order to gain an alive insight into the laws and harmonies of geometrical space.

BOOKS

L.Locher-Ernst	:	Urphänomene der Geometrie	1932
		Projektive Geometrie	1940
		Raum und Gegenraum	1957
G.Adams	:	Strahlende Weltgestaltung	1933
H.Baravalle	:	Das Reich geometrischer Formen	1935
L.Cremona	:	Projective Geometry. Oxford Press	1913
E.Bindel	:	Harmonien im Reiche der Geometrie	1964
G.Unger	:	Das offenbare Geheimnis des Raumes	1965

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page

(* or "Harmonic" - the conventional term.
The author preferred "Harmonious" for this text.)

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PROJECTIVE GEOMETRY

The synthetic as opposed to analytic approach to projective geometry is independent of all calculation and measurement. The aim is to develop a qualitative approach that brings into play all the faculties of man.

" What can be grasped by thought has its effect in reality. What lives in the world comes to realisation in thought. "

(Quotation by E.Locher- Ernst from his " Space and Counterspace", a mathematical textbook on Projective Geometry.)

Science investigates the ideas behind or within the created world. If the fine arts were to concern themselves with expressing ideas, little difference would exist between ART and SCIENCE. Further,we must also emphasize the difference between Mathemetics and Science and especially the difference between GEOMETRY and NATURAL SCIENCES.
Since the elements of geometry are concepts ("ideas") not derived from sense perceptions, they can never be empirically investigated. Concepts in geometry are created in man's THOUGHT and IMAGINATION.

" Projective Geometry: a boundless domain of countless fields where reals and imaginaries, finites and infinites, enter on equal terms, where the spirit delights in the artistic balance and symmetric interplay of a kind of conceptual and logical counterpoint - an enchanted realm where thought is double and flows throughout in parallel streams." (C.J.Keyser)

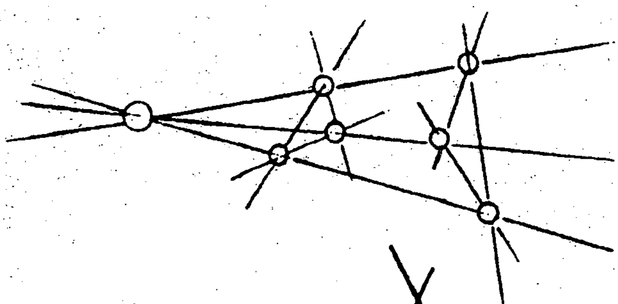
" In the house of mathematics there are many mansions and of these the most elegant is projective geometry. The beauty of its concepts, the logical perfection of its structure and its fundamental role in geometry recommend the subject to every student of mathematics."
(Morris Kline 1955)

S P A C E is a C O N C E P T .

Its elements are PLANES, POINTS and STRAIGHT LINES.

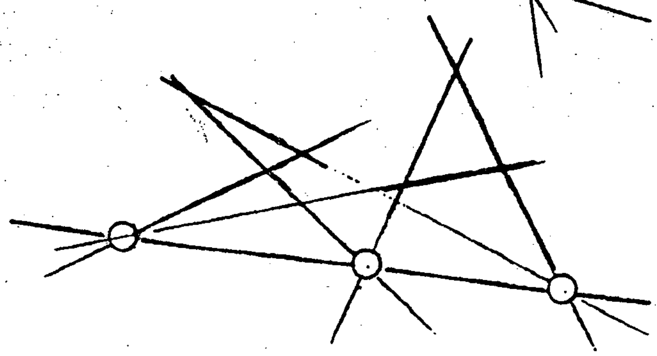
Projective Geometry introduces the elements at infinity. Two parallel lines meet at infinity in a point. Geometry takes this as a concept and finds no contradictions in spite of the fact that tactile experience can only remain in the finite.

A line has one and only one point at infinity.
A plane has one and only one straight line at infinity.
There is one and only one plane at infinity.
Points, lines and planes are characterised by their intersections and connections, without employing rigid definitions.
POLARITY is the main law underlying all projective geometry. CONNECTING - INTERSECTING are the two polar opposite activities.
A plane is either a field of points or a field of lines. The point is the polar opposite to the line in the geometry of the plane. The pointwise aspect is familiar to us all. The linewise aspect is less familiar and needs working at to be further understood.



Two perspective triangles (points are considered).

Corresponding points are in pairs on three concurrent lines - i.e. three lines through a centre.



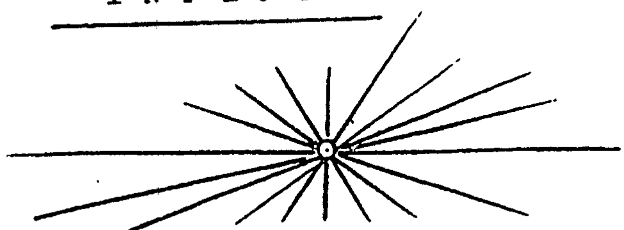
Two perspective trilaterals (sides are considered).

Corresponding sides intersect in pairs in three collinear points - i.e. three points on an axis.

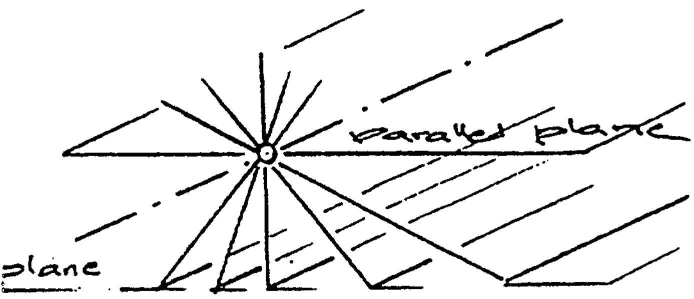
Theorem by Desargues (1593-1662)

If two triangles are perspective with respect to a point, they are also perspective with repect to a line and vice versa. (The proof follows later under "Five Planes")

INFINITY



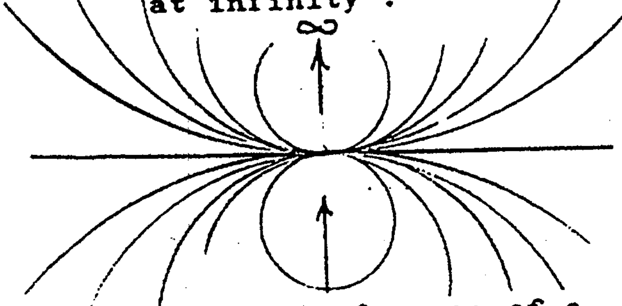
Between a "pencil"of lines passing through a point and a "range"of points lying on a line is a one-to-one correspondence. To each ray (line) corresponds one and only one point. To the ray parallel to the range corresponds the point at infinity of the range.



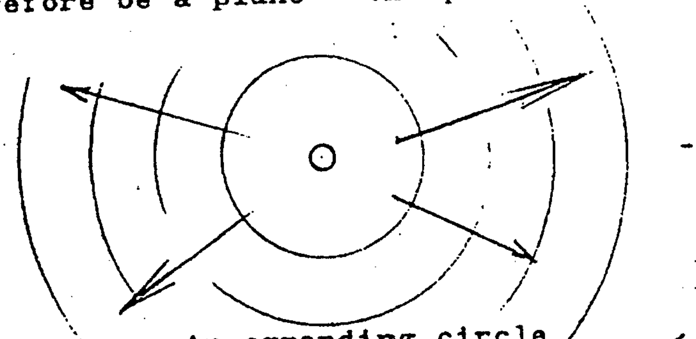
In the field of lines every line has a point at infinity. The sum of these points must make up a straight line- the line at infinity- for a form which contains one point from each line, can only be a straight line.

There is only one plane at infinity -

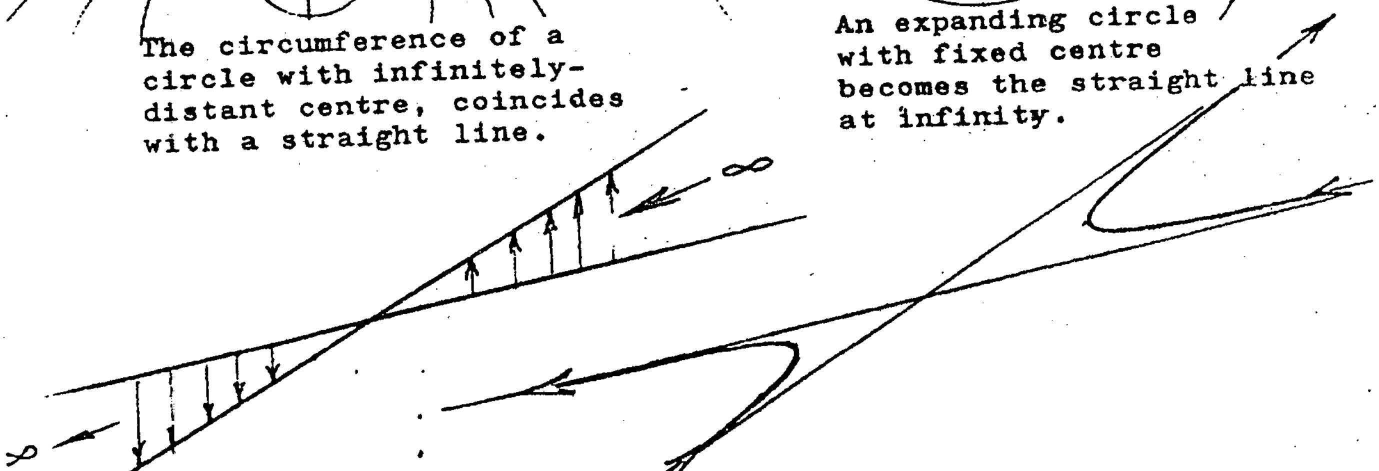
Two planes intersect in a straight line. A plane intersects all other planes in space in straight lines. Every plane has one and only one straight line at infinity. The sum of these lines at infinity must therefore be a plane - the plane at infinity .



The circumference of a circle with infinitely-distant centre, coincides with a straight line.



An expanding circle with fixed centre becomes the straight line at infinity.



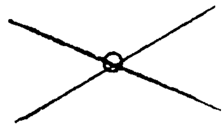
All the points on a line are in continuous sequence. At the overstepping of infinity a change takes place. This is seen only in relation to the environment of the line. Diagrammatically expressed above (left): as the point of the vertical arrow must remain on the same line, the arrow switches to the other side of the line after it has passed over infinity. The other example (right) shows a curve passing over infinity twice and returning on the opposite sides of the two lines.

(∞ = symbol for infinity)

Lines are straight and infinitely extended.

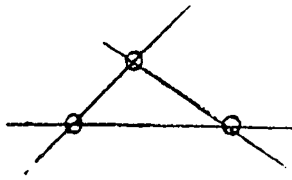
Points have no extension and are infinitely small.

In the GEOMETRY of the PLANE :

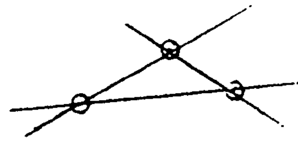


2 lines determine 1 point

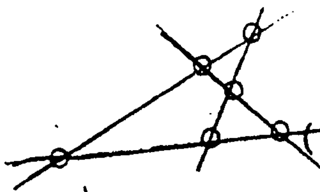
2 points determine 1 line



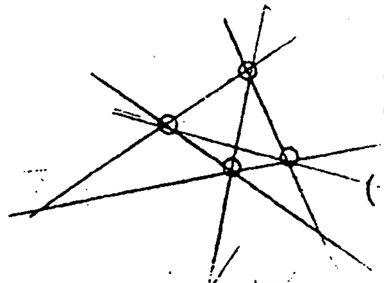
3 lines determine 3 points



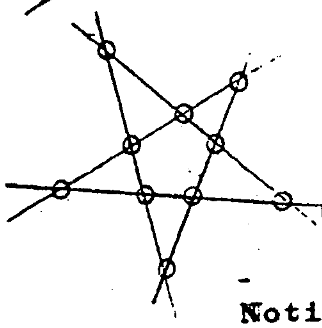
3 points determine 3 lines



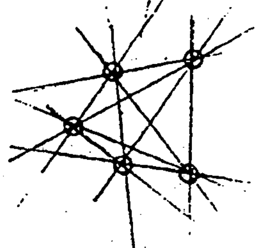
4 lines determine 6 points (previous 3 + 3)



4 points determine 6 lines (previous 3 + 3)

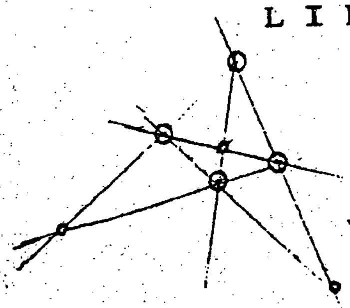


5 lines determine 10 points (previous 6 + 4)

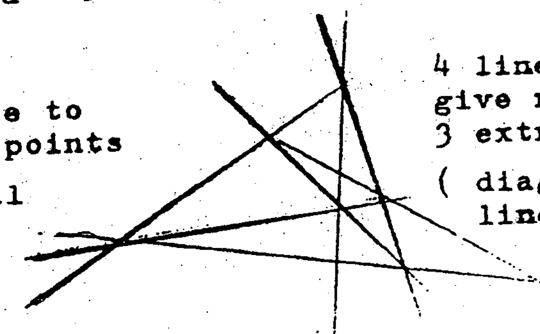


5 points determine 10 lines (previous 6 + 4)

Notice the POLARITY in the plane between LINES and POINTS.



4 points give rise to 3 extra points (diagonal points)



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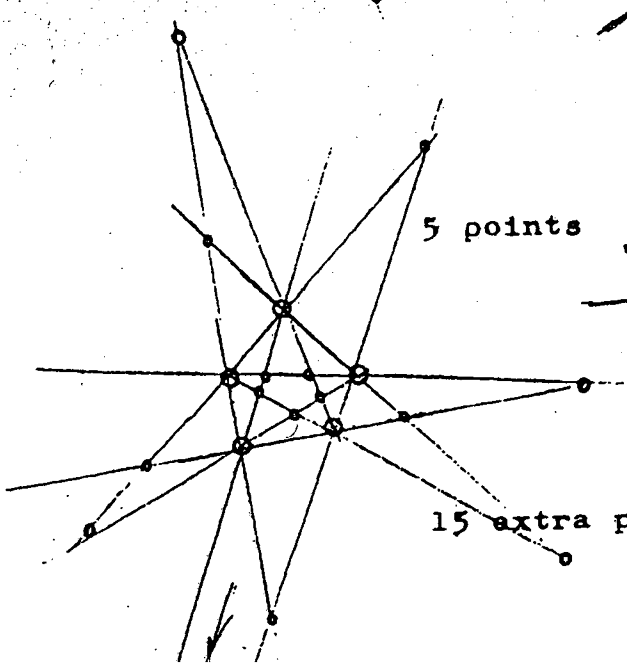
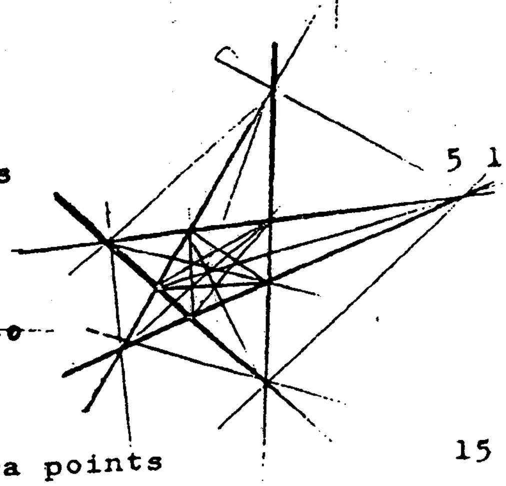


5 points

15 extra points

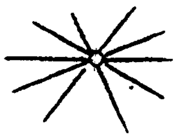


5 lines

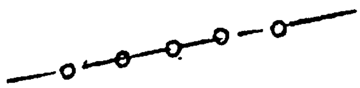
15 extra lines

Basic CONFIGURATIONS between POINT, LINE and PLANE.

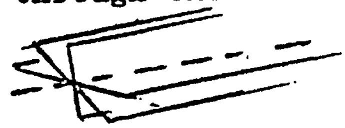
Pencil of lines: (in a plane) all the lines through one point



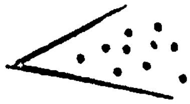
Range of points: all the points on one line



Pencil of planes: all the planes through one line

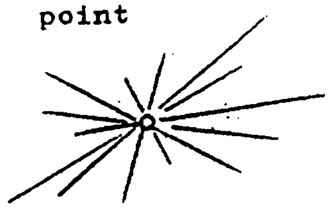


Field of points: all the points in a plane

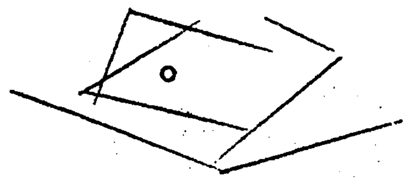


Field of lines: all the lines in a plane

Bundle of lines: (in space) all the lines through one point



Bundle of planes: all the planes through one point



In space the polar opposite of a line is another line.

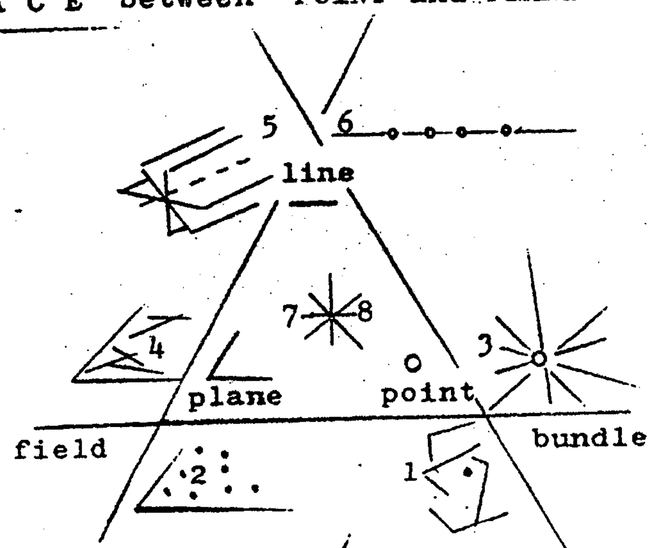
POLARITY in SPACE between POINT and PLANE

1 planes through a point
2 points in a plane

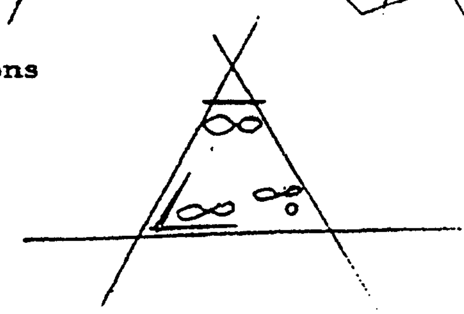
3 lines through a point
4 lines in a plane

5 planes through a line
6 points on a line

7 lines through a point in a plane --
8 lines in a plane through a point (self-polar)



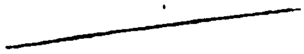
Consider the above configurations with point, line or plane at infinity.



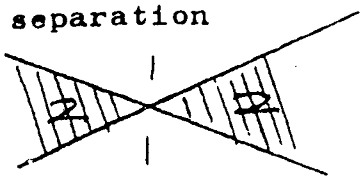
Space can be considered as a Space of points
a Space of planes
or a Space of lines.

P O L A R I T Y in the P L A N E POINT —— LINE

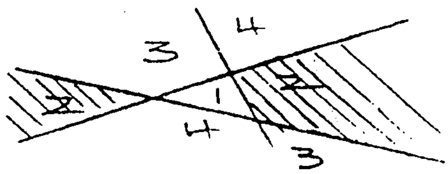
Lines separate a field of points into SECTIONS



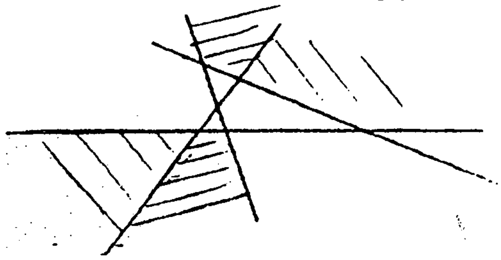
One line creates no separation



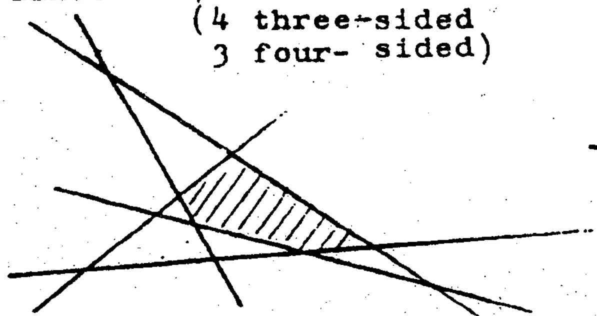
2 lines -- 2 sections



3 lines -- 4 three-sided sections



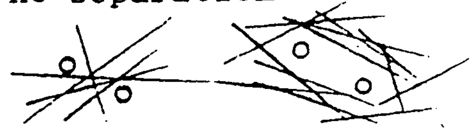
4 lines -- 7 sections (4 three-sided 3 four- sided)



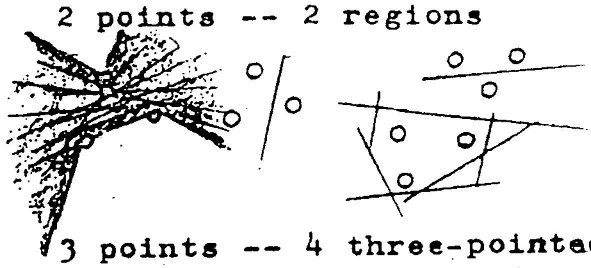
5 lines -- 11 sections (one five-sided)

Points separate a field of lines into REGIONS

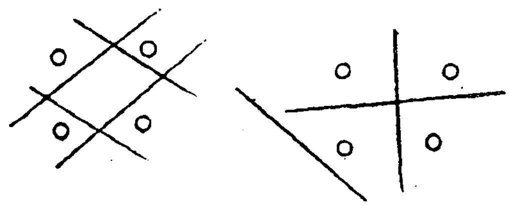
One point creates no separation



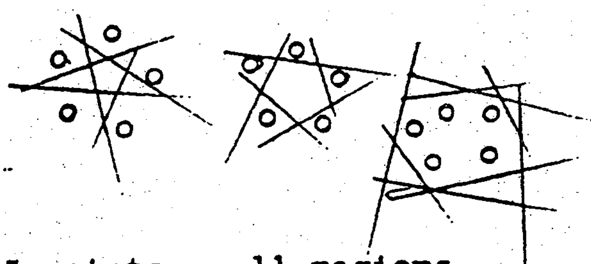
2 points -- 2 regions



3 points -- 4 three-pointed regions



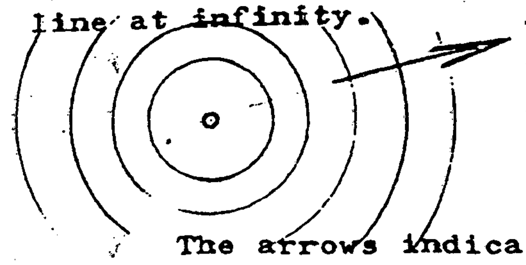
4 points -- 7 regions (4 three-pointed 3 four- pointed)



5 points -- 11 regions (one five-pointed)

Regions overlap each other — to represent them with overlapping colours gives further clarity. Regions and sections are complementary.

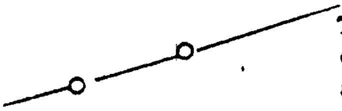
In the field of points the smallest circle is the point, the largest is the area bounded by the line at infinity.



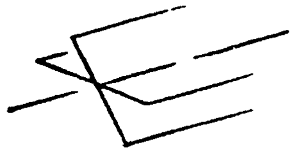
In the field of lines the smallest circle is the line at infinity; the largest circle includes all the lines in the plane.

The arrows indicate the growth of the circles.

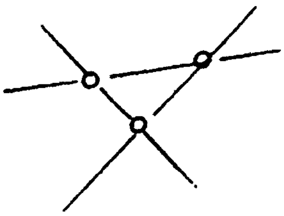
In the geometry of space, point and plane are polar opposite.



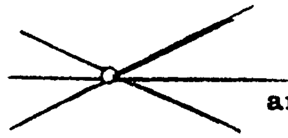
Two points determine a line.



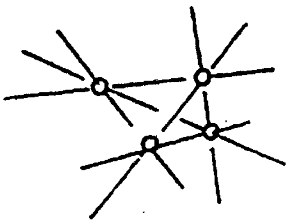
Two planes determine a line.



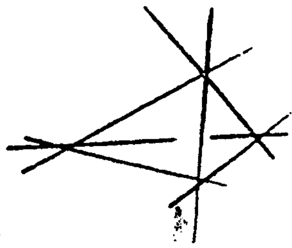
Three points determine three lines and one plane.



Three planes determine three lines and one point.



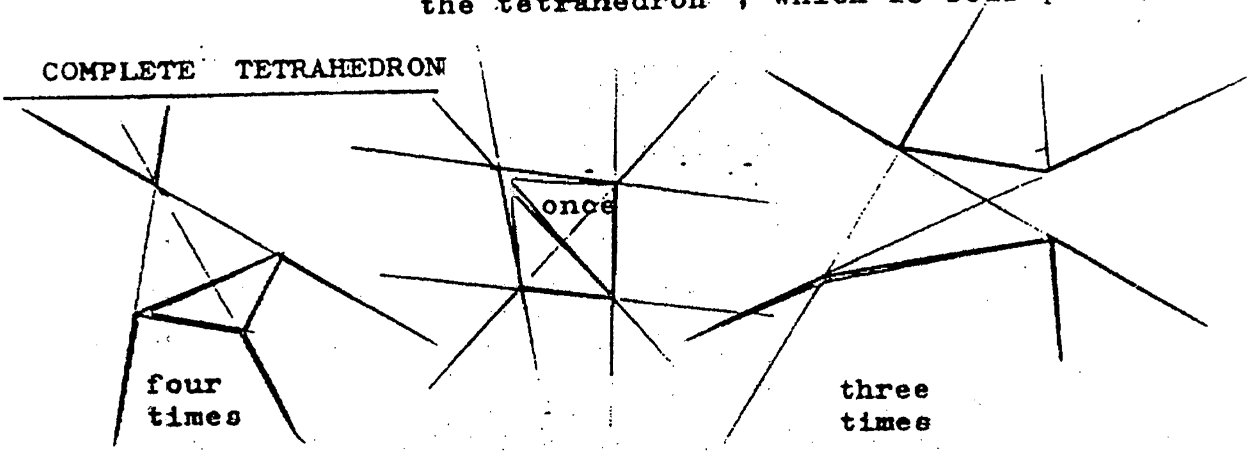
Four points determine six lines and four planes.



Four planes determine six lines and four points.

Four points or four planes both produce the tetrahedron, which is self-polar.

COMPLETE TETRAHEDRON



The complete tetrahedron separates space into eight tetrahedra.

In the PLANE :

								polar:
points	1	2	3	4	5	6	7	lines
lines	-	1	3	6	10	15	21	points
extra points	-	-	-	3	15			extra lines

In SPACE:

											polar:
points	1	2	3	4	5	6	7	8	9	10	planes
lines	-	1	3	6	10	15	21	28	36	45	lines
planes	-	-	1	4	10	20	35	56	84	120	points

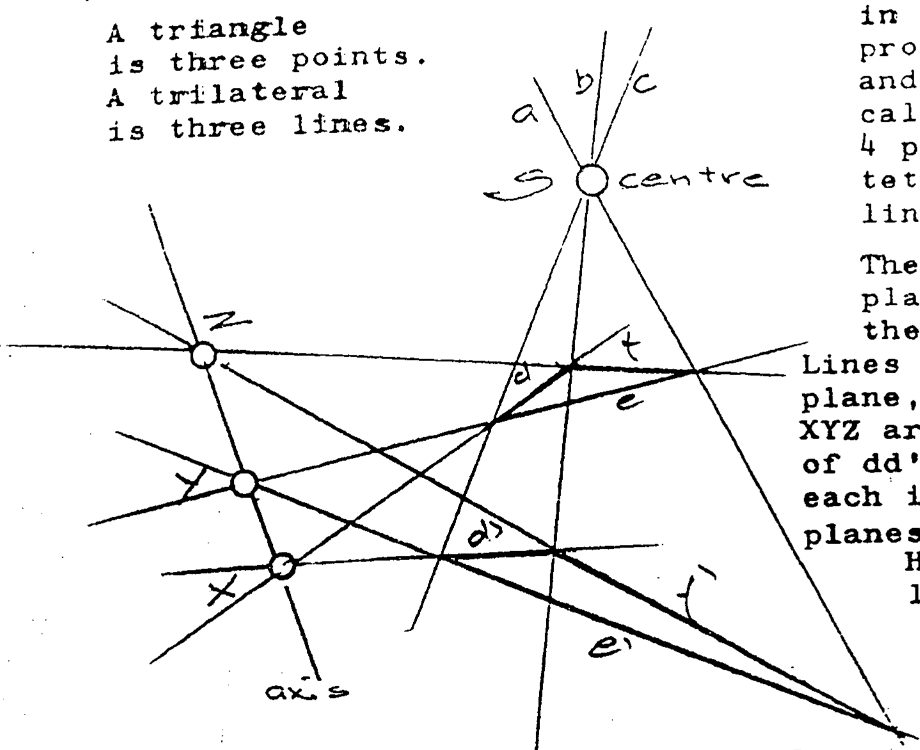
F I V E   P L A N E S

Two perspective triangles at the same time determine two perspective trilaterals.

A triangle is three points. A trilateral is three lines.

The configuration of five planes establishes Desargues' theorem (see page 3).

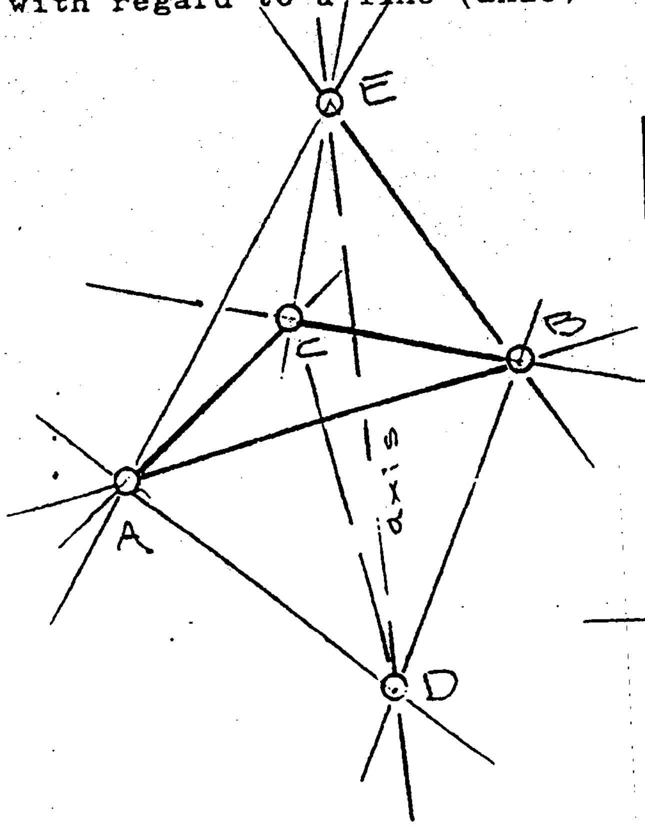
2 planes intersect in a line. 3 planes produce 3 lines abc and one point S called a three-flat. 4 planes produce a tetrahedron with 6 lines and 4 points.



The 4th and the 5th planes intersect on the axis. Lines def are in the 4th plane, d'e'f' in the 5th. XYZ are intersections of dd' ee' ff' and are each in the 4th and 5th planes. Hence they must lie on the axis. Every point can be considered as a centre S and every line as an axis, to a particular pair of perspective triangles. This drawing is a projection from an eye point onto the plane of the paper. The theorem of Desargues is therefore also proved for the plane: If two triangles are perspective with regard to a point (centre), they are also perspective with regard to a line (axis). In the plane the theorem is self-polar.

F I V E   P O I N T S

in space produce the polar configuration to the five planes.



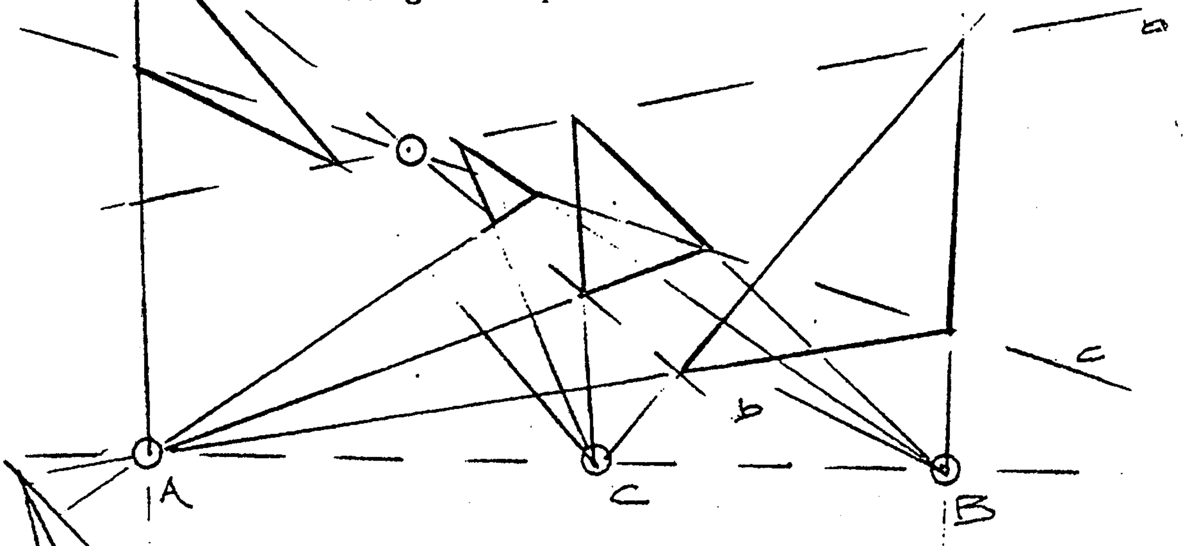
Two three-flats from D and E are perspective with regard to the plane ABC. Corresponding edges: AD -AE , BD -BE, CD -CE, connect in planes through the axis DE. 5 points produce 10 lines and 10 planes. 5 planes produce 10 lines and 10 points.

The theorem of Desargues can be considered to be the basic law underlying all projective geometry.

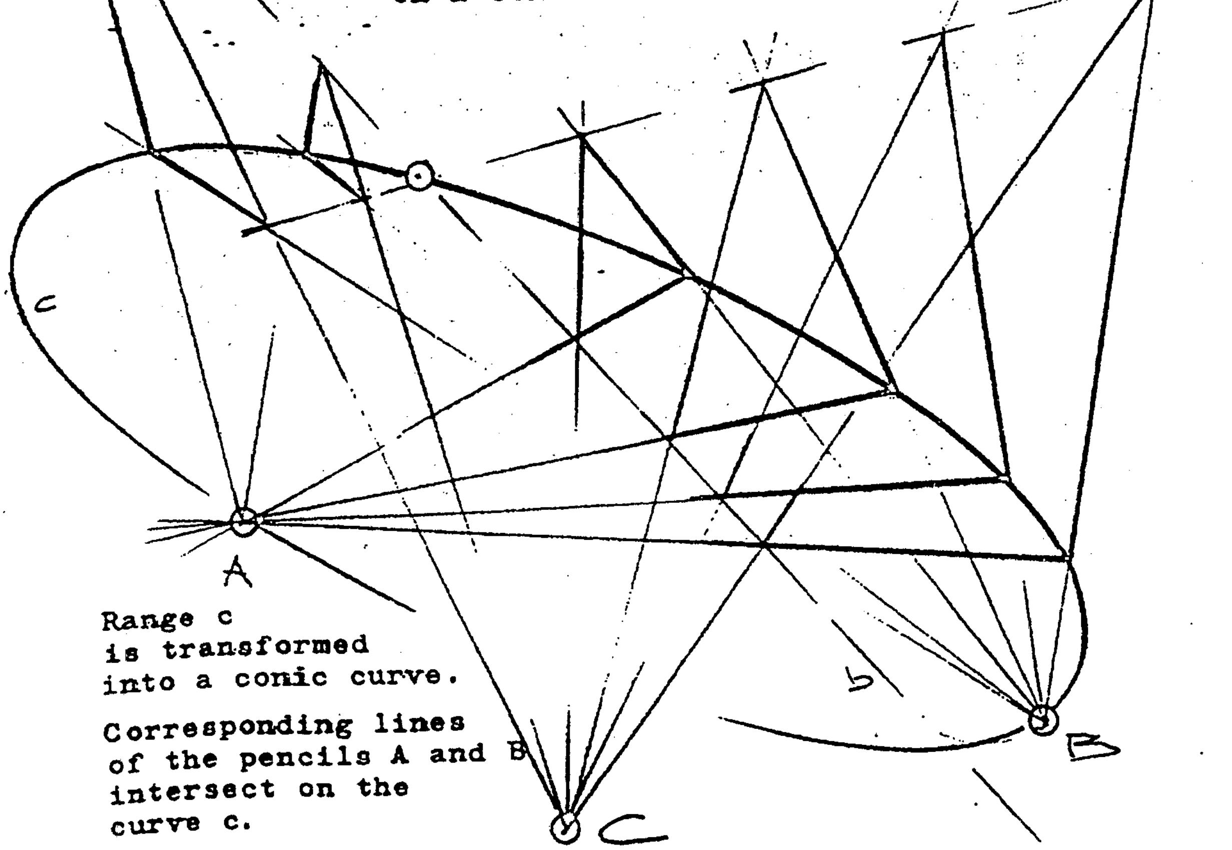
From perspective triangles with a common axis to the CONIC CURVE

Pencils of lines (in ABC) on a line.

Ranges of points (on abc) through a centre.



With point C no longer on the line AB, sides of the triangles intersect on a conic curve.



Range c is transformed into a conic curve.

Corresponding lines of the pencils A and B intersect on the curve c.

Draw in various ways - especially with centre and/or axis at infinity.

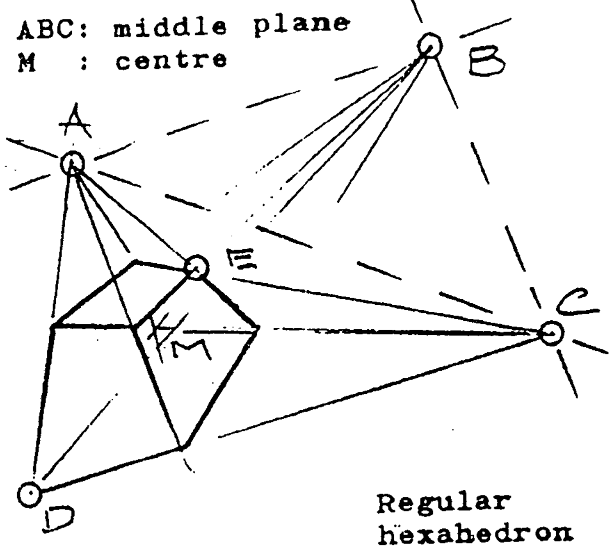
Also draw the polar opposite configuration (curve from tangents).

Five points determine a HEXAHEDRON

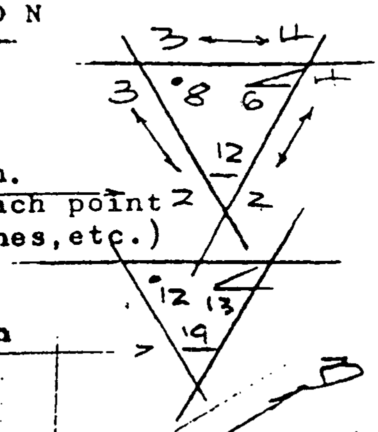
ABCD (tetrahedron) + E

ABC: middle plane

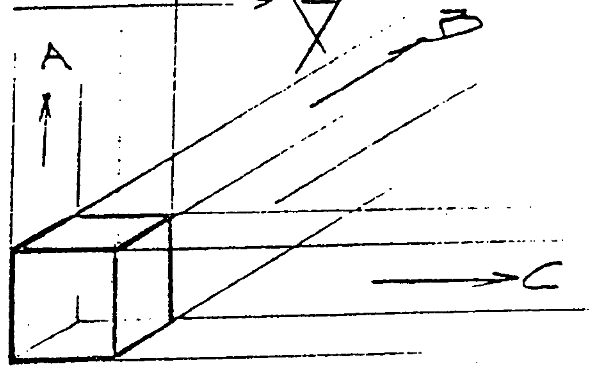
M : centre



Formula for a simple hexahedron. (though each point three lines, etc.)



Complete hexahedron



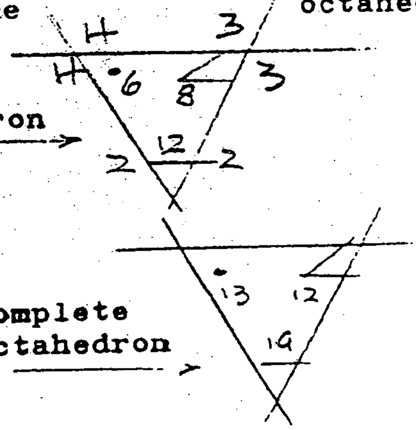
Regular hexahedron (cube). Middle plane at infinity.

Five planes determine an OCTAHEDRON

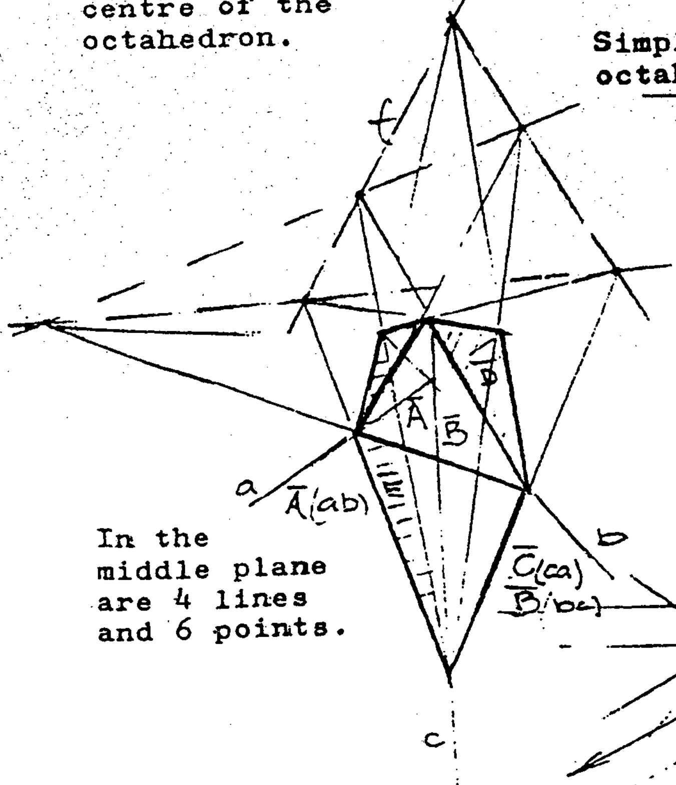
Planes ab($\overline{A}$), bc($\overline{B}$), ca($\overline{C}$). Plane $\overline{A}$ (i.e. through lines ab) and planes $\overline{B}$(bc) and $\overline{C}$ (ca) are diagonal through the centre of the octahedron.

Planes $\overline{D}$ $\overline{E}$ intersect in f (Desargues' configuration) and are opposite faces of the octahedron.

Simple octahedron

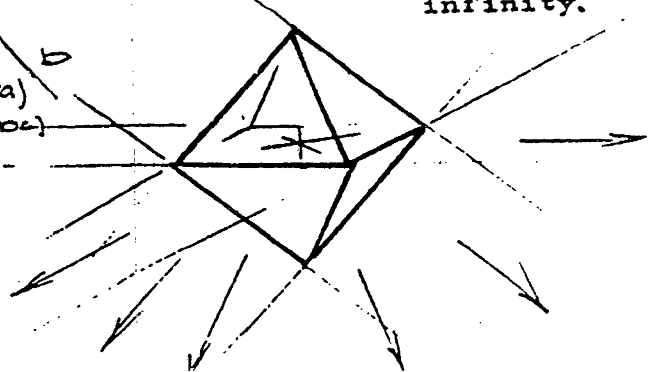


Complete octahedron



In the middle plane are 4 lines and 6 points.

Regular octahedron: 4 lines and 6 points at infinity.



*H A R M O N I O U S   F O U R   C O N F I G U R A T I O N

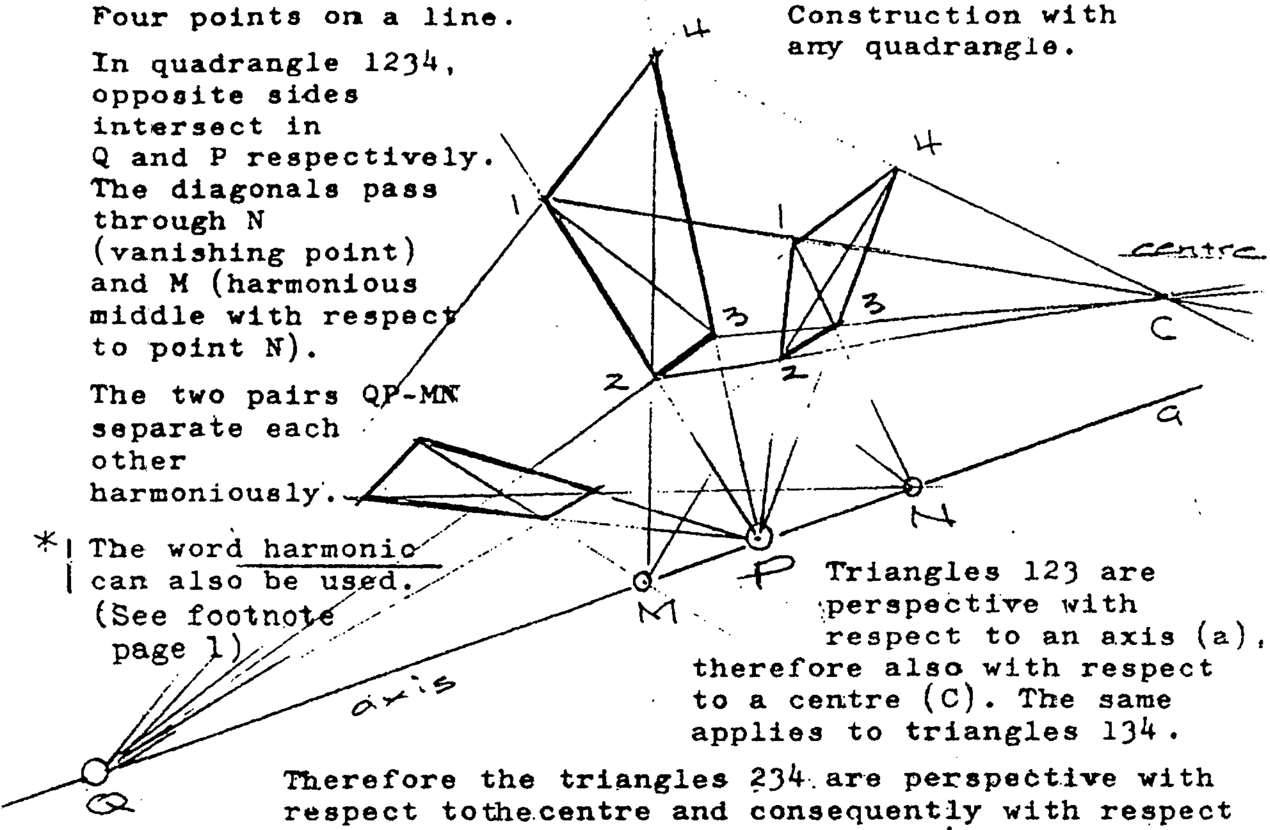
Four points on a line.

In quadrangle 1234, opposite sides intersect in Q and P respectively. The diagonals pass through N (vanishing point) and M (harmonious middle with respect to point N).

The two pairs QP-MN separate each other harmoniously.

*| The word harmonic can also be used. (See footnote page 1)

Construction with any quadrangle.

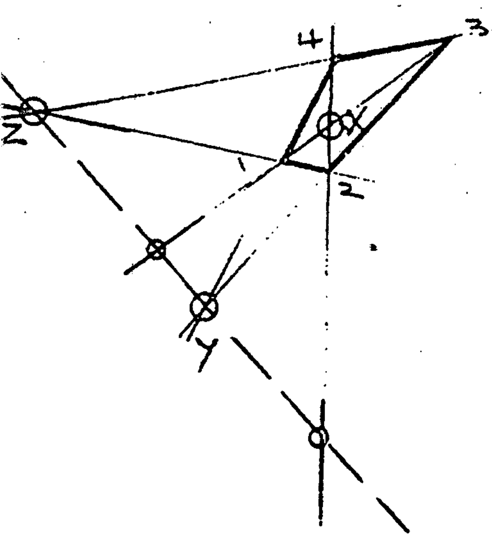


Triangles 123 are perspective with respect to an axis (a), therefore also with respect to a centre (C). The same applies to triangles 134.

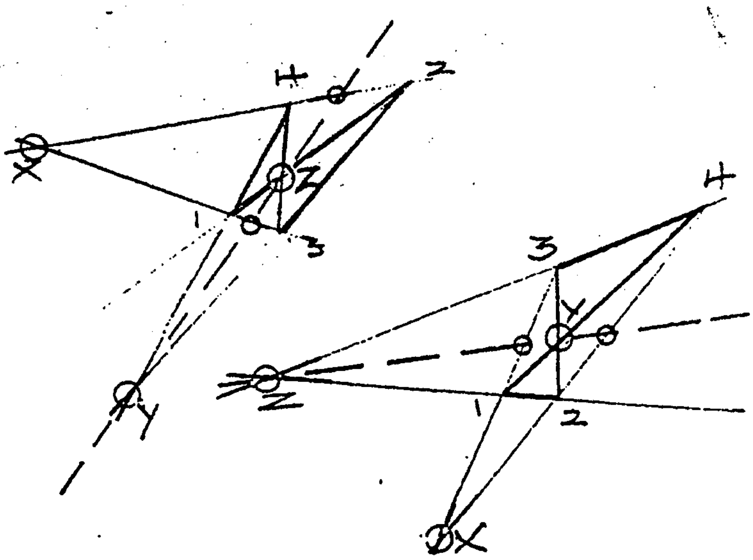
Therefore the triangles 234 are perspective with respect to the centre and consequently with respect to the axis. Hence the diagonals 24 always go through the same point M.

Draw various quadrangles from the same points PQN.

An axis (yz) as described, with four harmonious points, is shown in each figure below. There are three variations for forming a quadrangle in the finite.

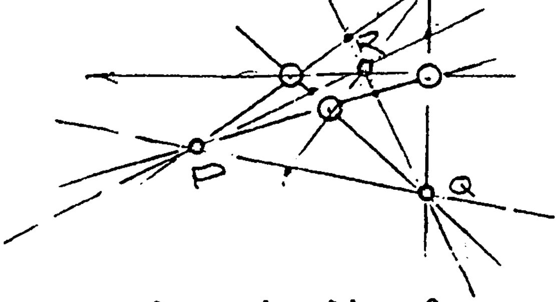


X, Y, Z are an "extra" triangle.



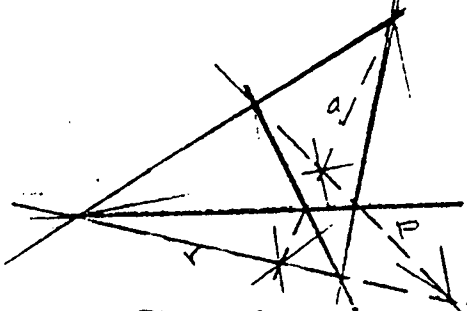
The polar opposite construction produces harmonious four rays. The drawing is self-polar.

the EXTRA TRIANGLE
of the quadrangle:
the intersections PQR
of the not yet intersected
6 sides.



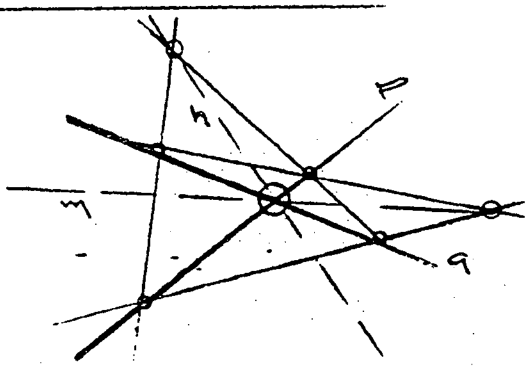
On each side of
the extra triangle are
four harmonious points.

the EXTRA TRILATERAL
of the quadrilateral :
the connections pqr
of the not yet connected
6 points.



Through each corner of the
extra trilateral are
four harmonious rays.

HARMONIOUS FOUR RAYS (lines)



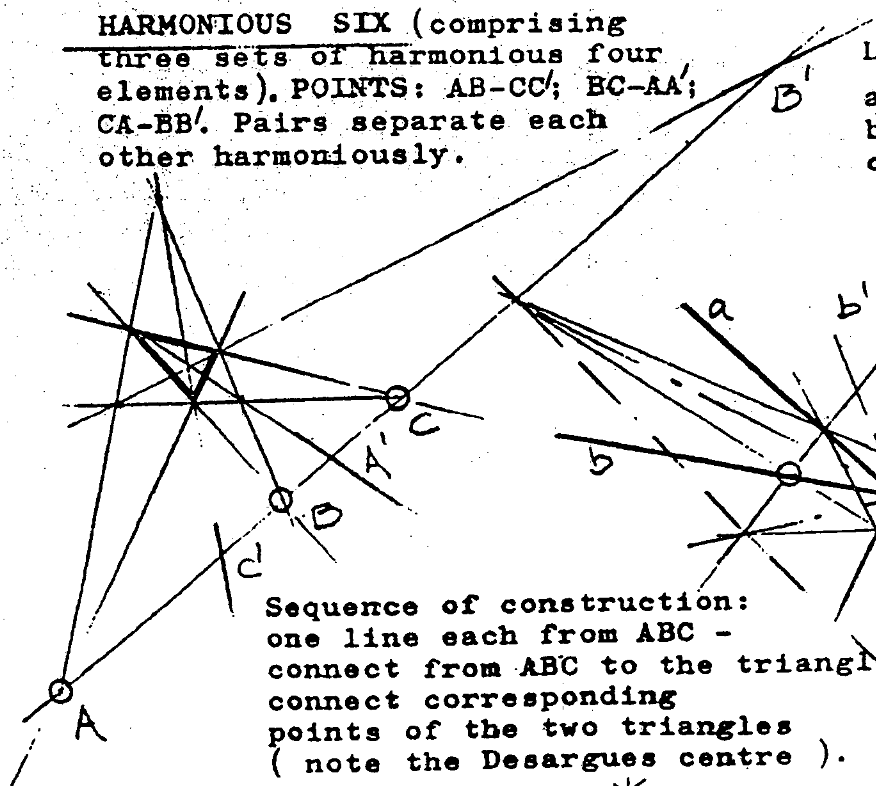
Harmonious four rays
through a point are polar to
harmonious four points on a line.

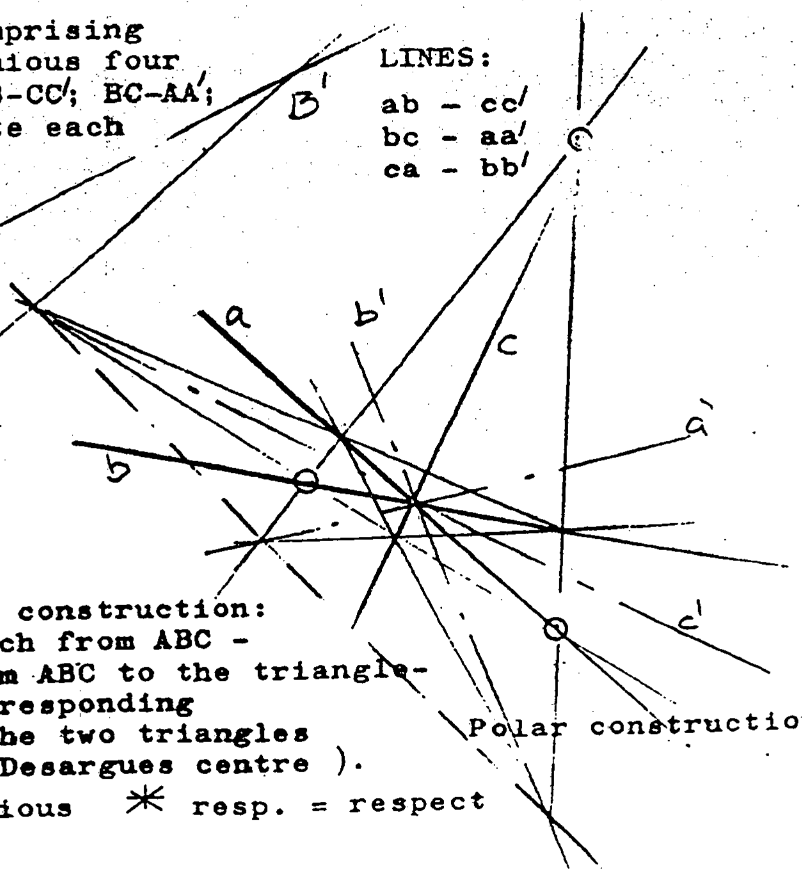
Opposite points of the
quadrilateral are on q and p.
The extra points are on
n (vanishing line) and on
m (harm.* middle line
with resp.* to n).
q p - m n separate each
other harmoniously.

HARMONIOUS SIX

(comprising three sets of harmonious four elements). POINTS: AB-CC'; BC-AA'; CA-BB'. Pairs separate each other harmoniously.

LINES:
ab - cc'
bc - aa'
ca - bb'





Sequence of construction:
one line each from ABC -
connect from ABC to the triangle -
connect corresponding
points of the two triangles
(note the Desargues centre).

Polar construction

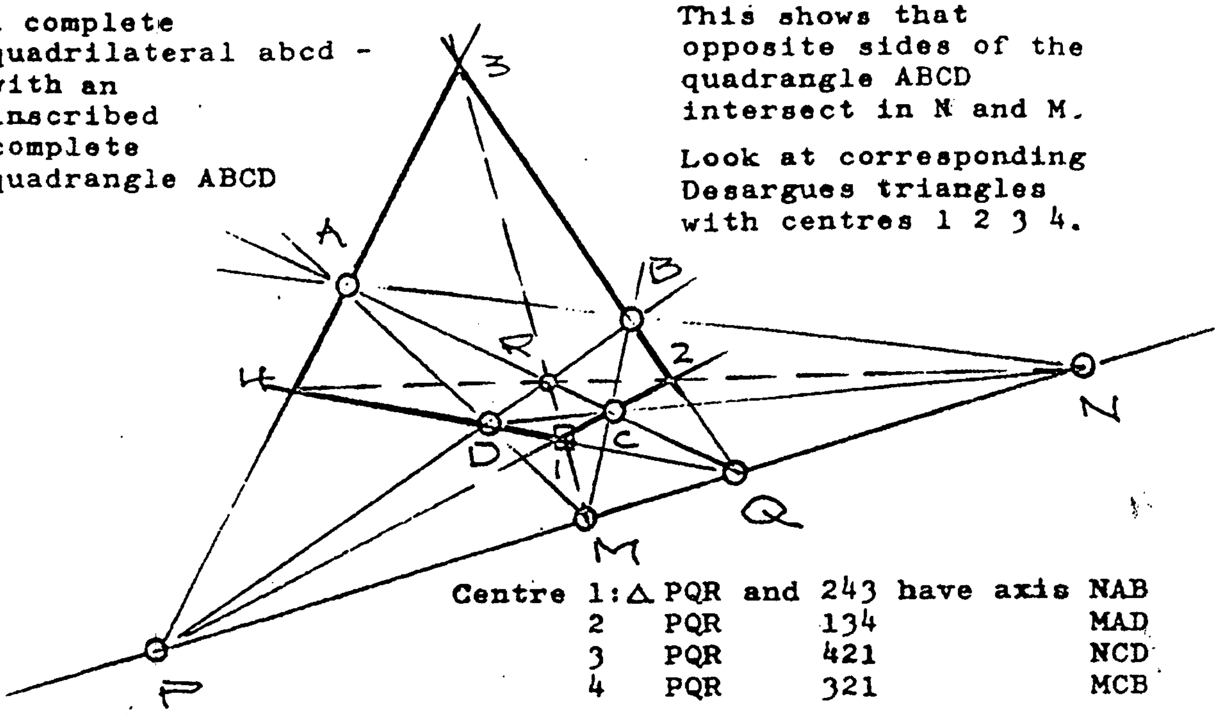
* harm. = harmonious * resp. = respect

FUNDAMENTAL HARMONIOUS CONFIGURATION in the plane.

A complete quadrilateral abcd - with an inscribed complete quadrangle ABCD

This shows that opposite sides of the quadrangle ABCD intersect in N and M.

Look at corresponding Desargues triangles with centres 1 2 3 4.



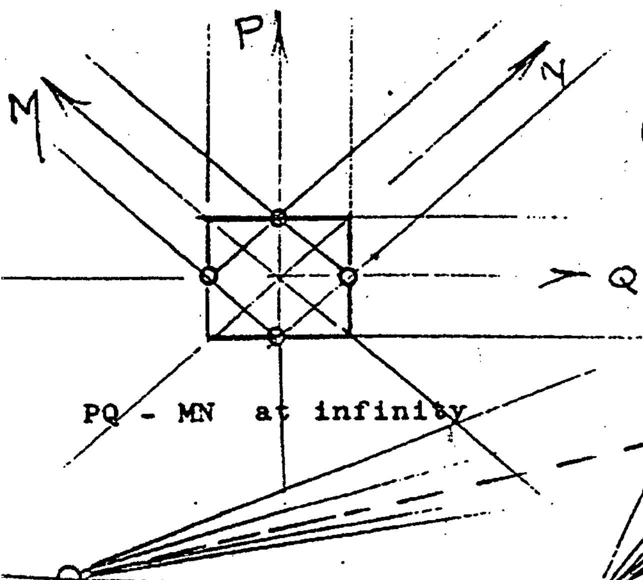
Centre 1: △	PQR	and 243	have axis	NAB
2	PQR	134		MAD
3	PQR	421		NCD
4	PQR	321		MCB

Triangle PQR is the extra triangle to abcd and the extra trilateral to ABCD.

Triangle MNR is the extra triangle to ABCD and the extra trilateral to abcd.

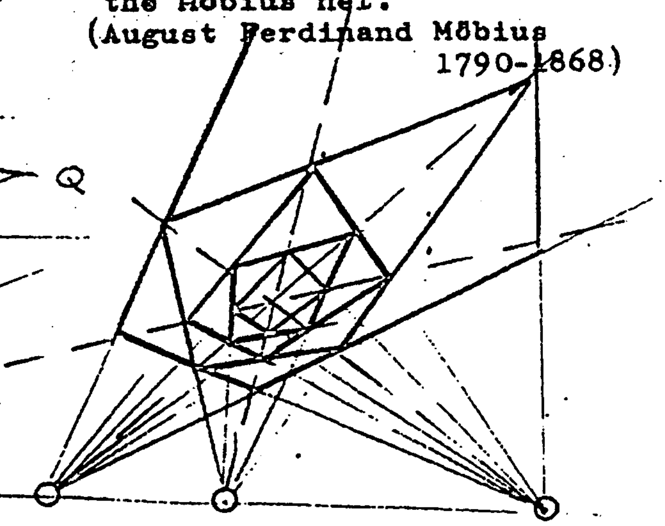
The configuration is self-polar (harmonious four rays). PQ - MN (the two pairs separating each other) can be reversed.

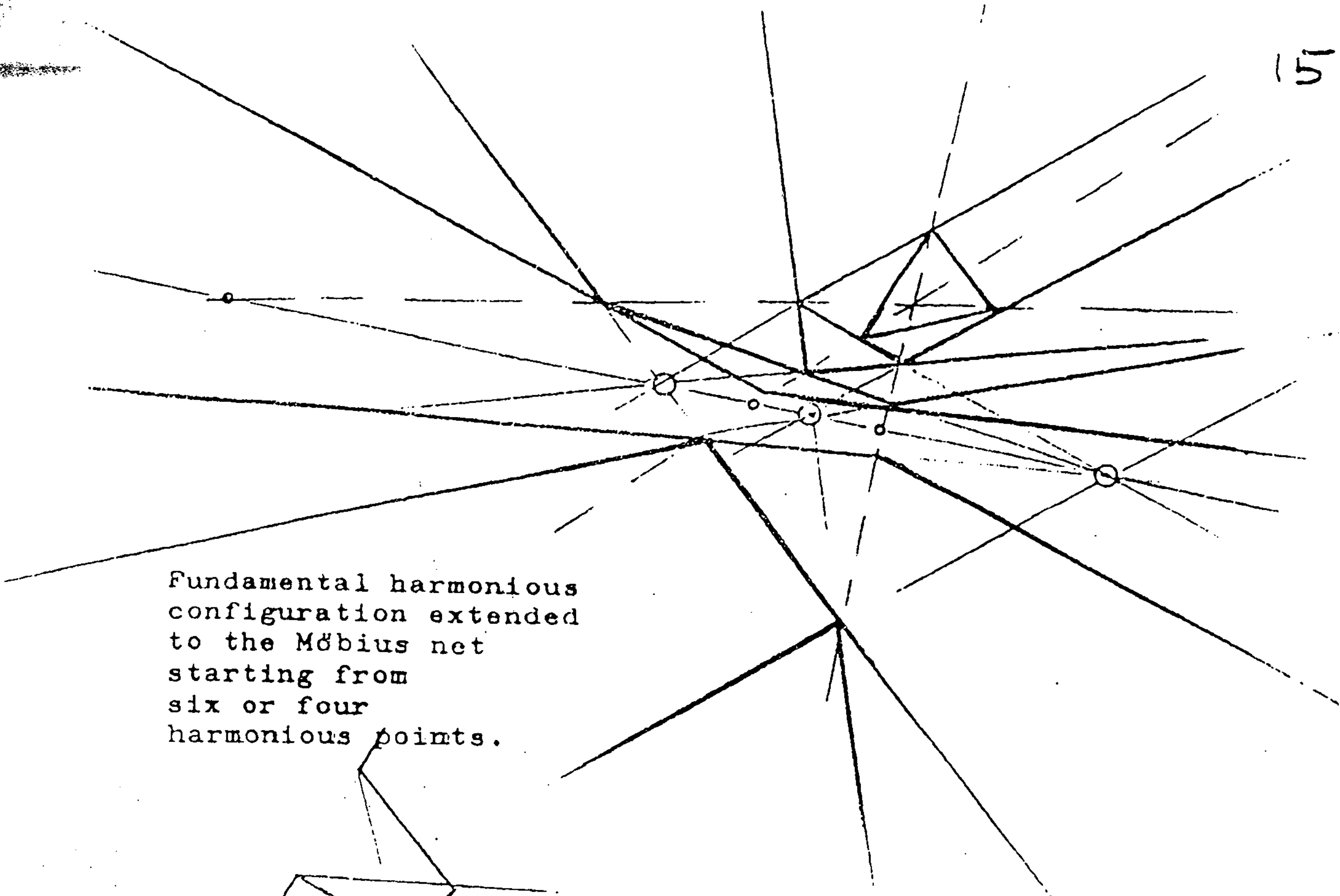
The harmonious relation is maintained by perspective- i.e. connections from any point to PQMN, or intersections of any line with four harmonious rays. (R is chosen anywhere.)



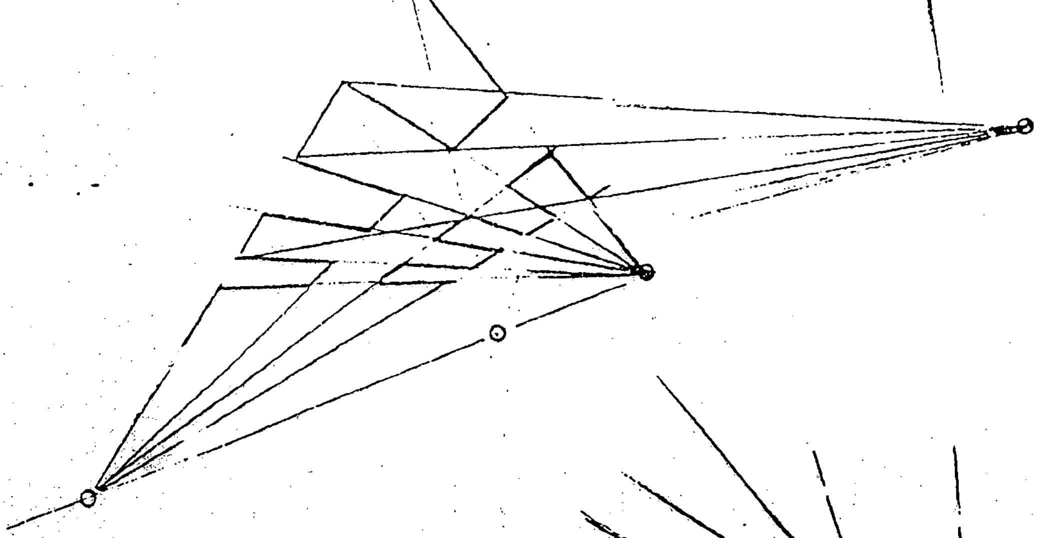
PQ - MN at infinity

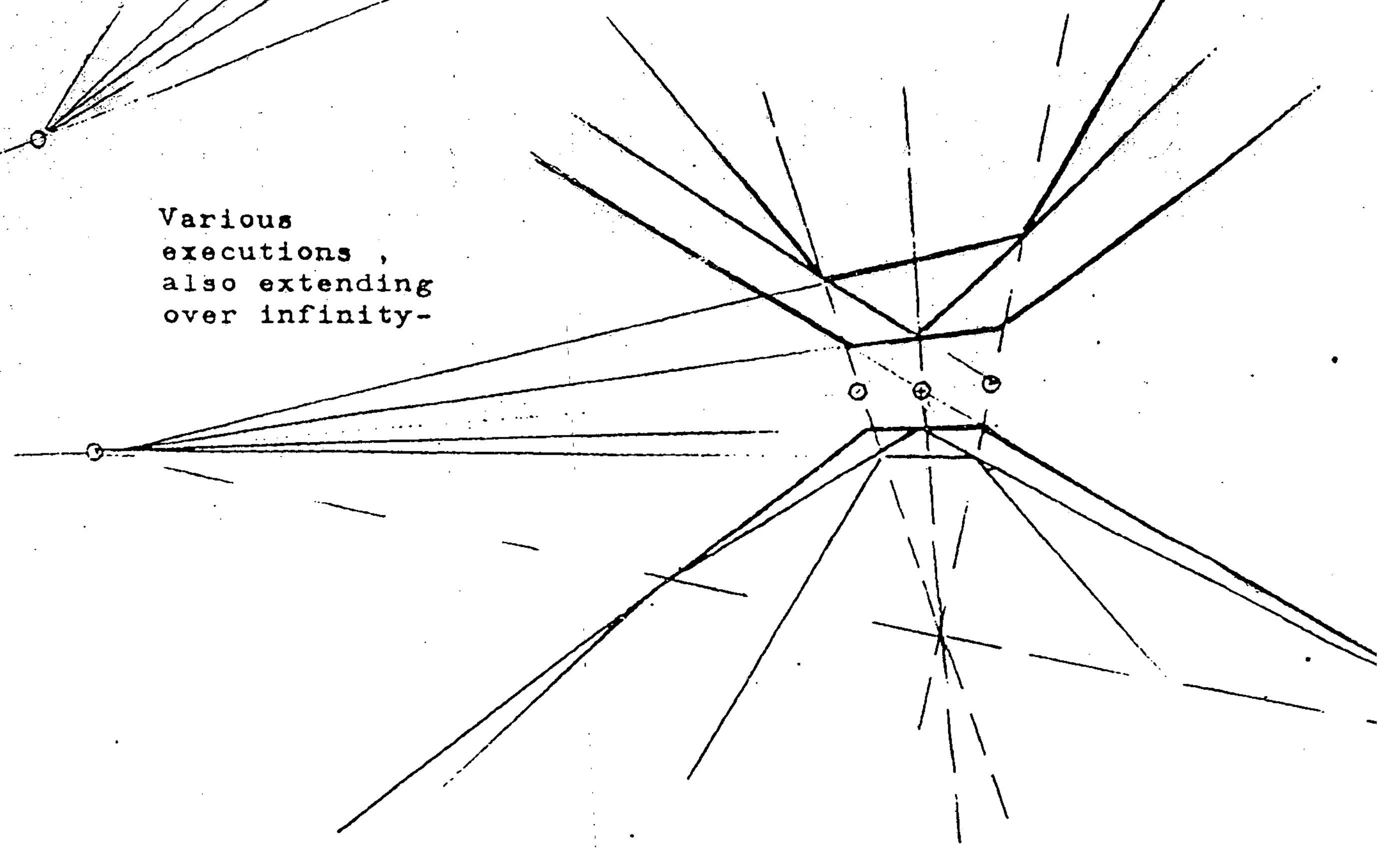
Circumscribing with quadrilaterals and inscribing with quadrangles produces the Möbius net.
(August Ferdinand Möbius 1790-1868)





Fundamental harmonious configuration extended to the Möbius net starting from six or four harmonious poimts.

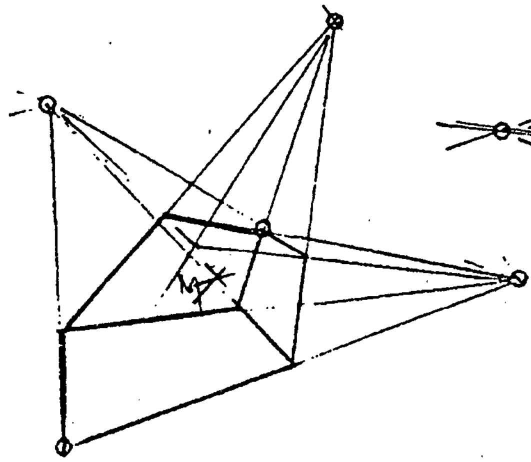




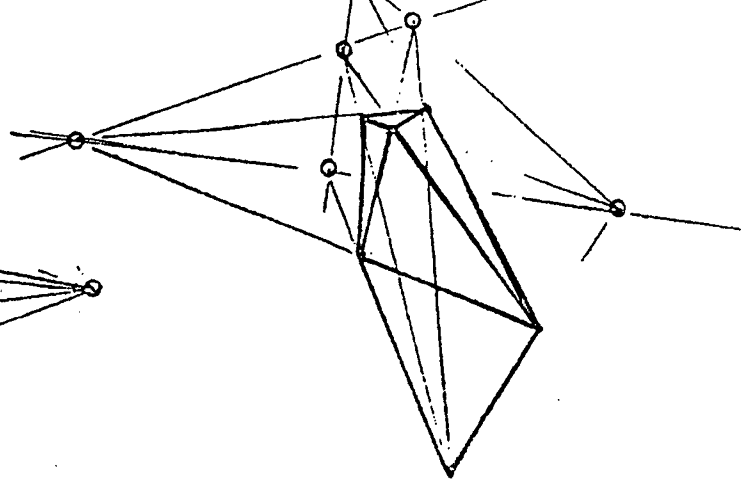
Various executions , also extending over infinity-

FUNDAMENTAL STRUCTURE of SPACE

A Hexahedron is determined by 5 points.



An Octahedron is determined by 5 planes.

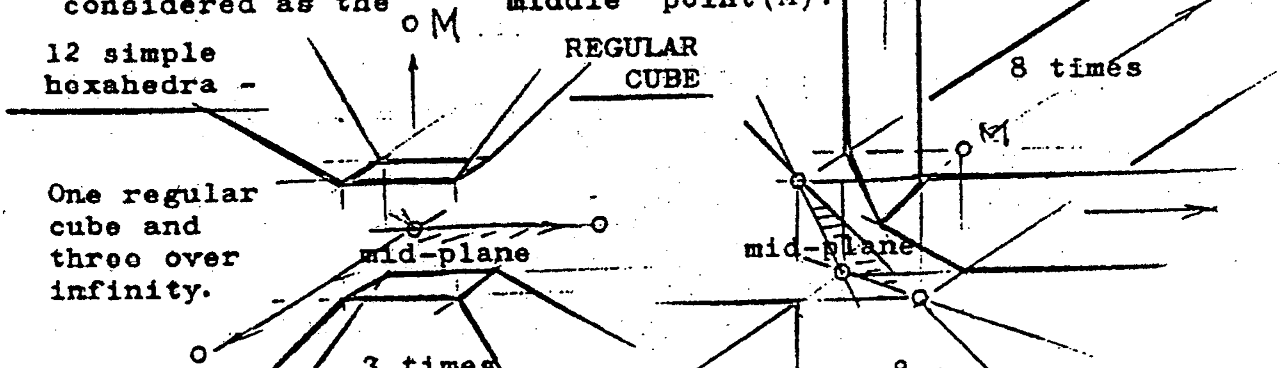


Both solids possess the same configuration-

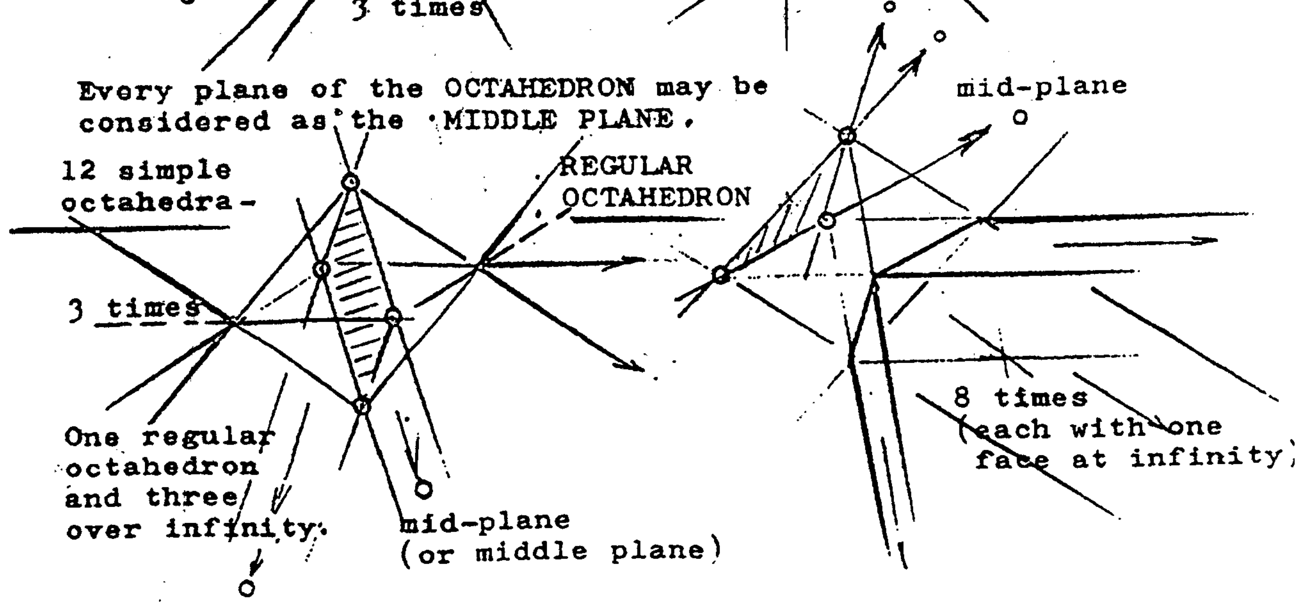
Hexahedron	8+4 points		Octahedron	6+6 points
	12+4 lines			12+4 lines
	6+6 planes			8+4 planes

Both are self-polar: on each line 3 points, in each plane 4 lines, through each point 4 lines.

Every point of the hexahedron may be considered as the middle point (M).



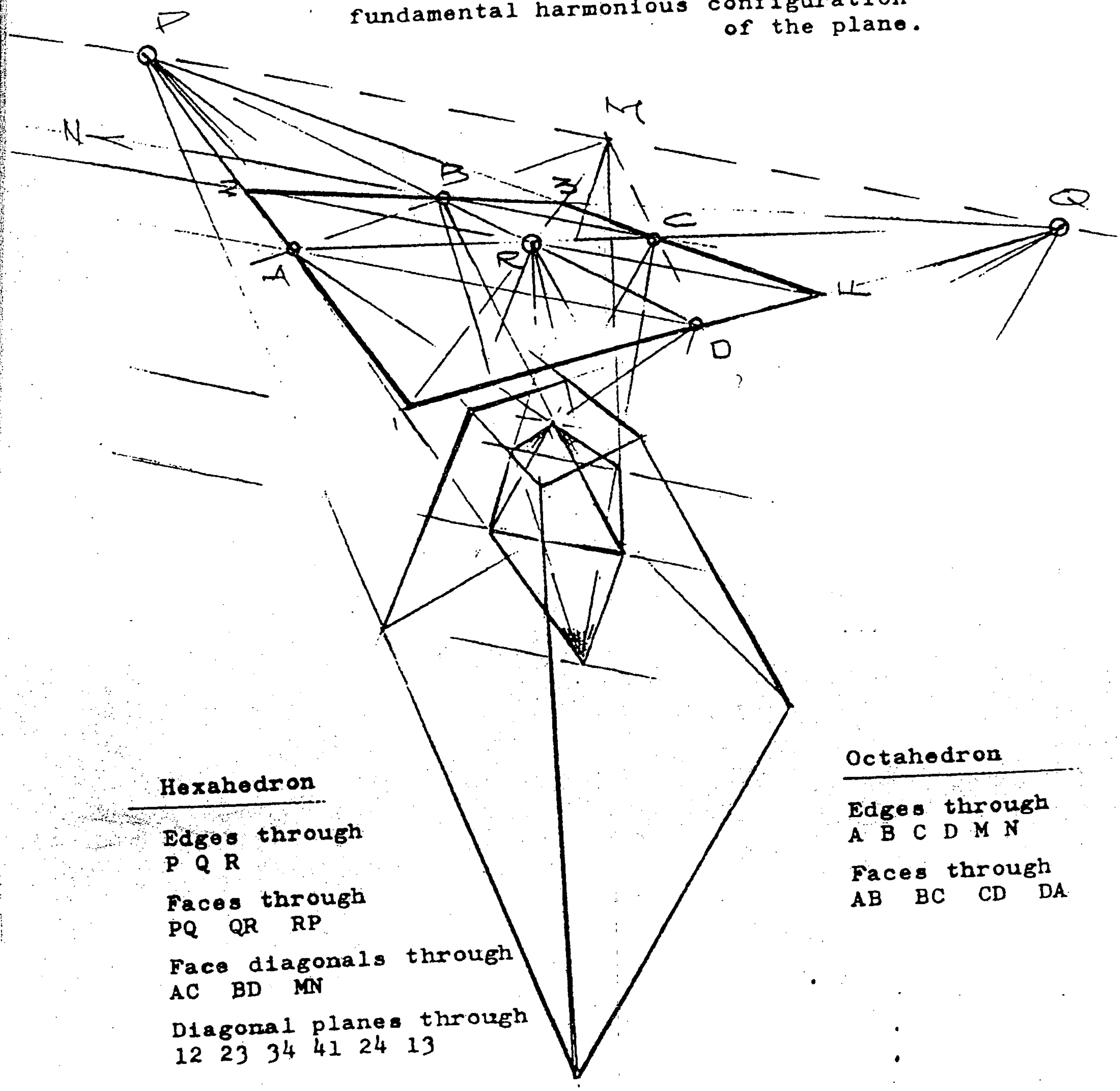
Every plane of the OCTAHEDRON may be considered as the MIDDLE PLANE.



FUNDAMENTAL HARMONIOUS CONFIGURATION of SPACE

Hexahedron with inscribed Octahedron.

In the middle plane is the fundamental harmonious configuration of the plane.



Hexahedron

Edges through
P Q R

Faces through
PQ QR RP

Face diagonals through
AC BD MN

Diagonal planes through
12 23 34 41 24 13

Octahedron

Edges through
A B C D M N

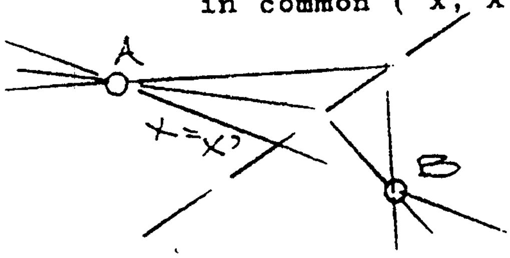
Faces through
AB BC CD DA

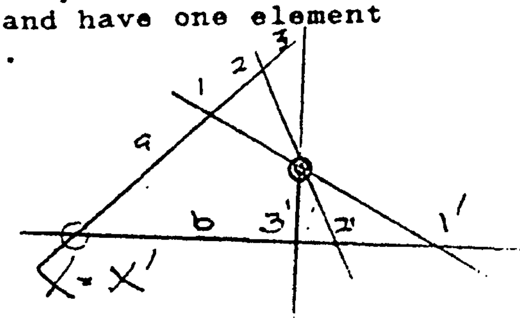
P E R S P E C T I V E

Two perspective pencils in points A and B are the connections with a range.

Two perspective ranges on lines a and b are the intersections with a pencil.

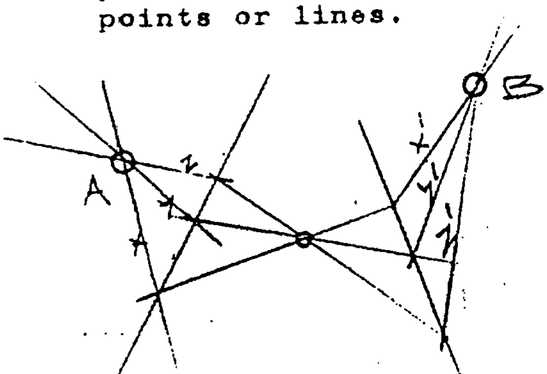
They are determined by two corresponding pairs of elements and have one element in common (x, X).

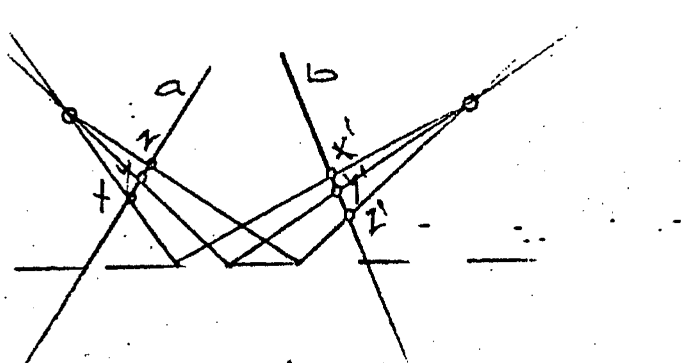




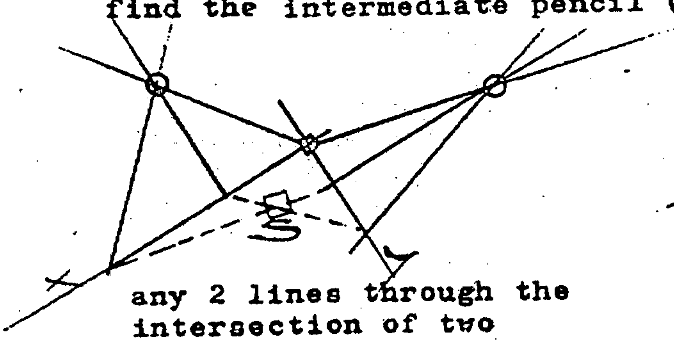
P R O J E C T I V I T Y is a chain of perspectives.

Two projective ranges or pencils are determined by three pairs of corresponding points or lines.

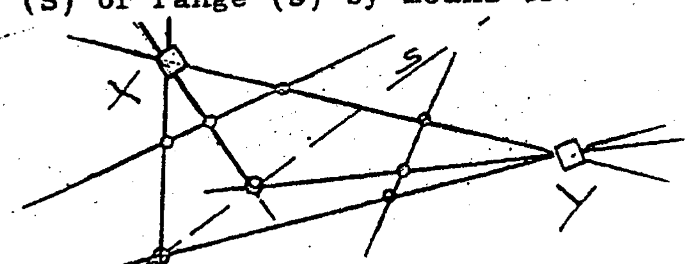




Three pairs of corresponding elements are given: find the intermediate pencil (S) or range (s) by means of:



any 2 lines through the intersection of two corresponding lines.



any 2 points at the connection of two corresponding points.

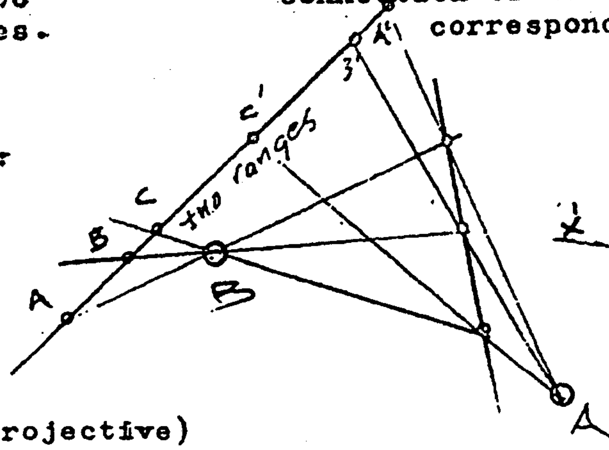
A projectivity in itself comprises :

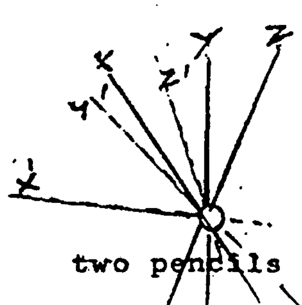
two ranges on the same base line

or

two pencils through the same base point.

(⊼ = symbol for projective)





Theorem of PAPPOS (circa 300 AD)

If a hexagon has its points on two straight lines (three points on each line) then the intersections of opposite sides are collinear.

Polar opposite :

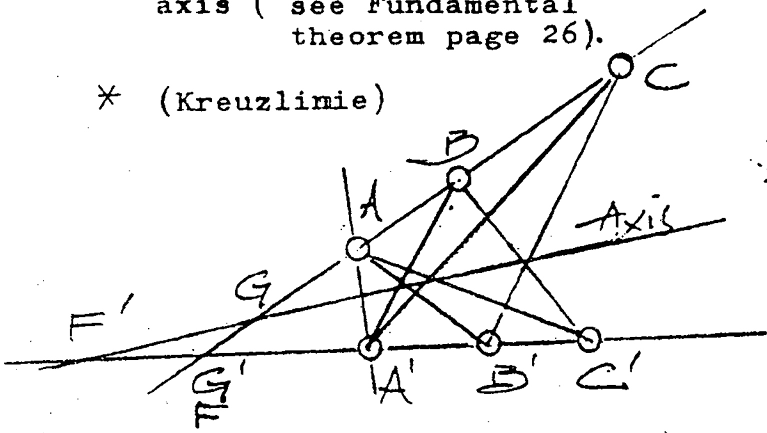
If a hexagram has its lines through two points (three lines through each point) then the connections of opposite points are concurrent.

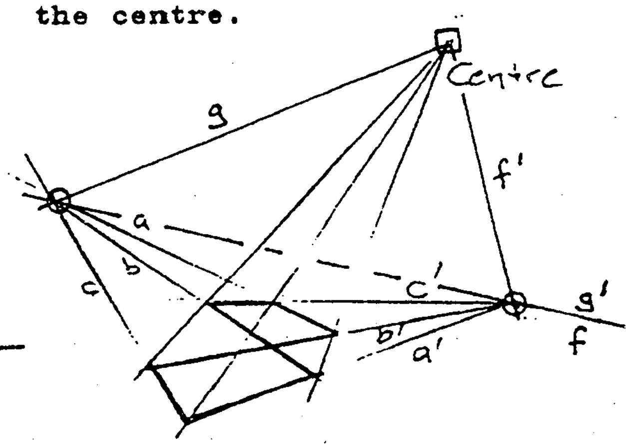
(This is a special case of Pascal's theorem -page 22)

Consider the six points as 3 pairs of projective points. Pencils from A A' are perspective as they have a line in common. The same goes for B B' and C C'.
* Cross lines intersect on the axis (see Fundamental theorem page 26).

* (Kreuzlinie)

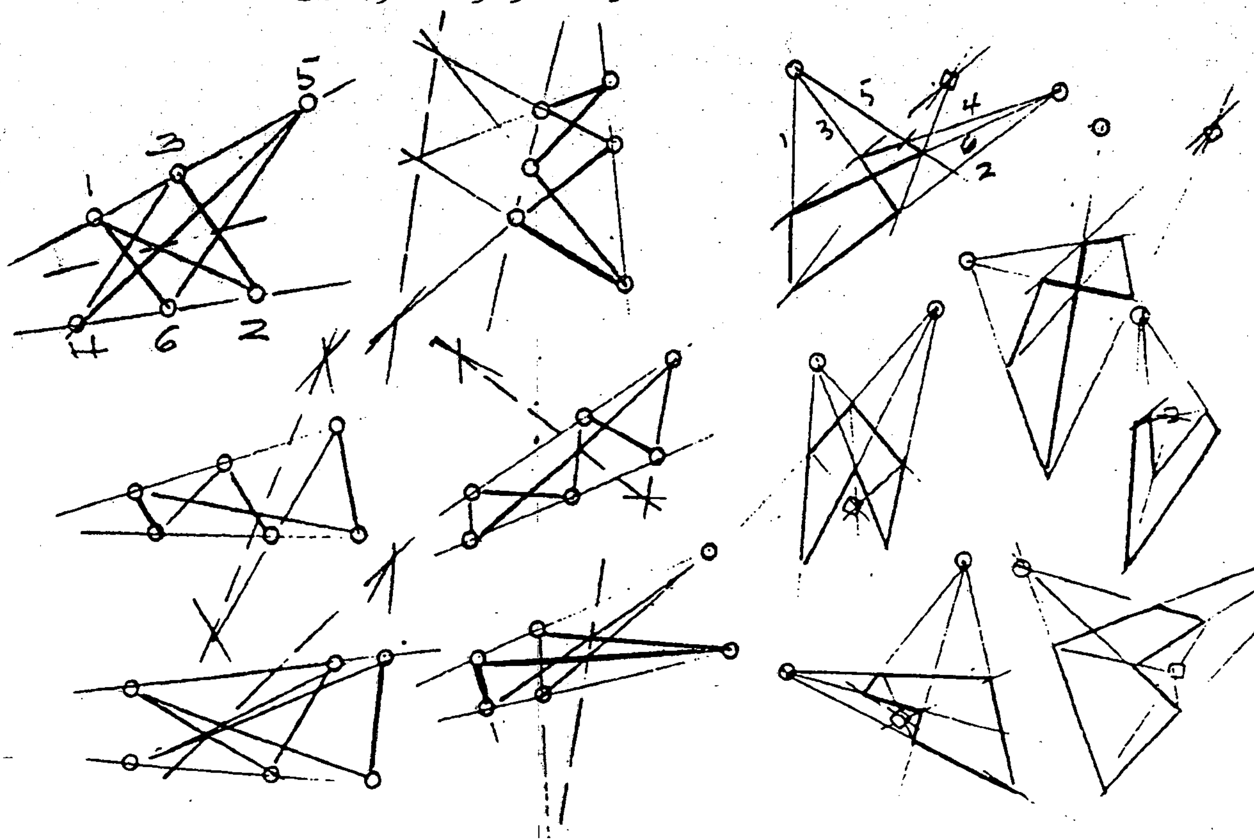
Consider the six lines as 3 pairs of projective lines. Ranges a a' are perspective as they have a point in common. The same goes for bb' and cc'.
Cross points connect through the centre.





There are six different ways to connect the 6 points or to intersect the 6 lines (in the finite).

To give numbers to the points / lines :
12-45 23-56 34-61 cross or connect.



THE CONIC CURVE

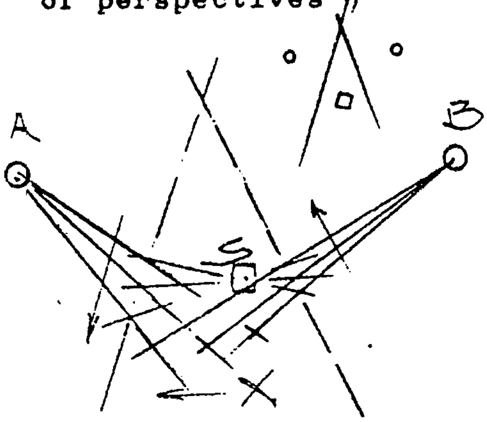
Pointwise construction

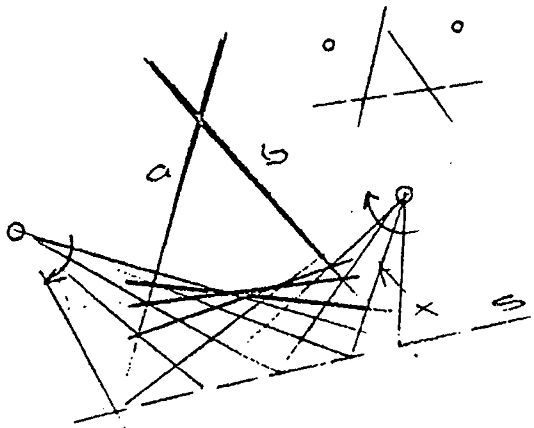
Corresponding lines of two projective pencils intersect on a conic.

Linewise construction (tangents produce envelope)

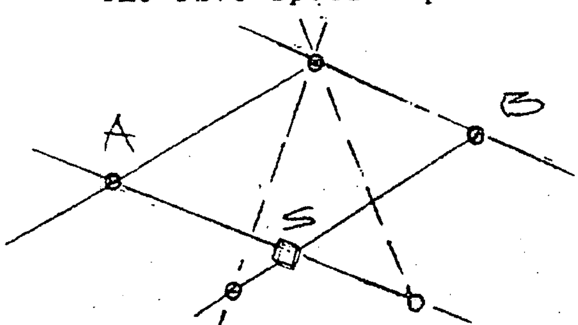
Corresponding points of two projective ranges connect on a conic.

Five points or five lines determine a conic curve. (Five elements are needed for the shortest chain of perspectives)



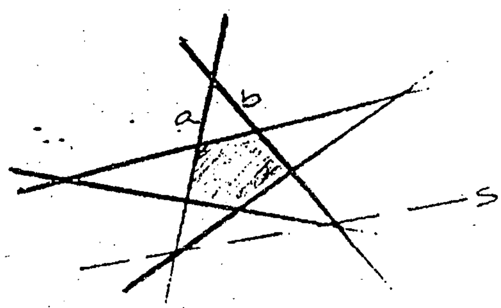


The five special points:

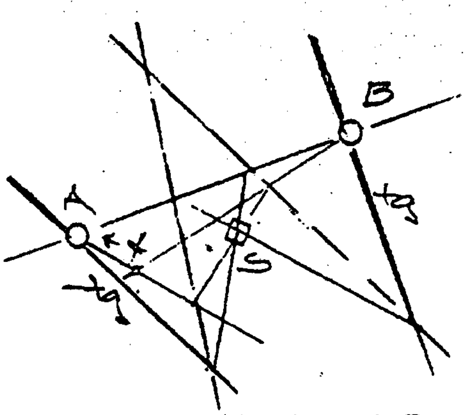


The conic is in the five-pointed region.

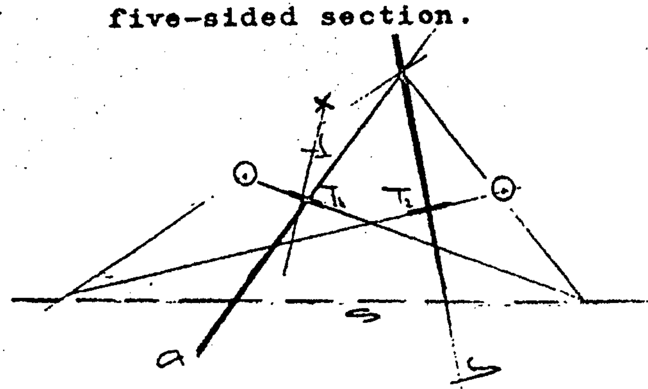
The five special tangents:



The conic is in the five-sided section.



Tangents in A and B are found when X meets A or B, and the chords XA, XB become tangents.

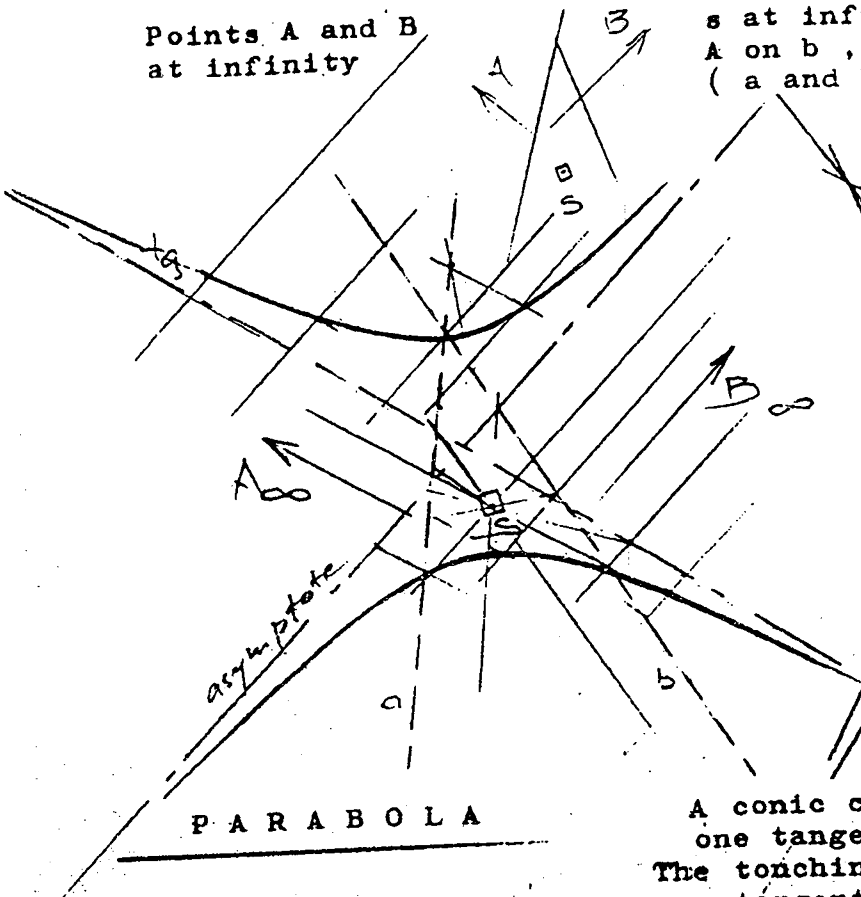


Points of contact on a and b are similarly found when tangent x coincides with a or b.

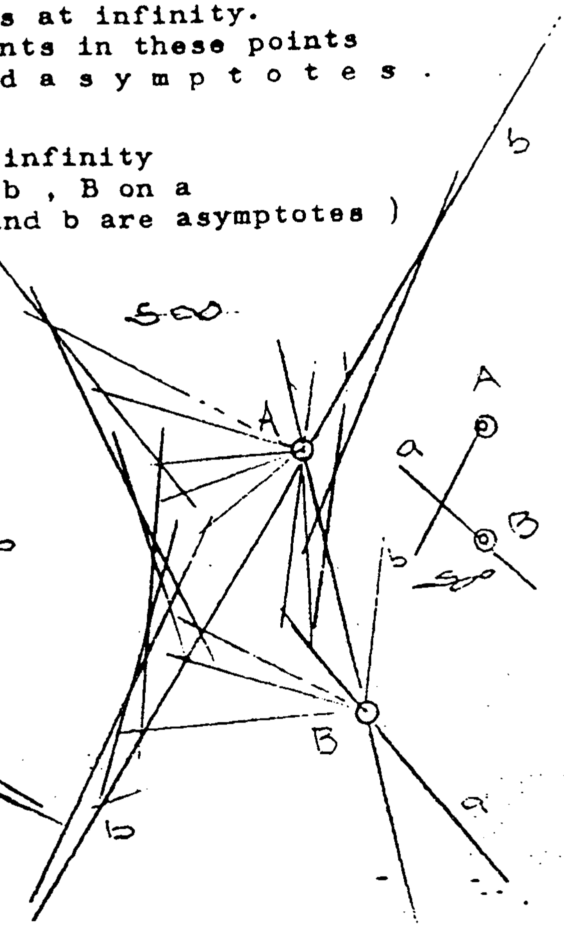
HYPERBOLA

A conic curve with two points at infinity. The tangents in these points are called a s y m p t o t e s .

Points A and B at infinity

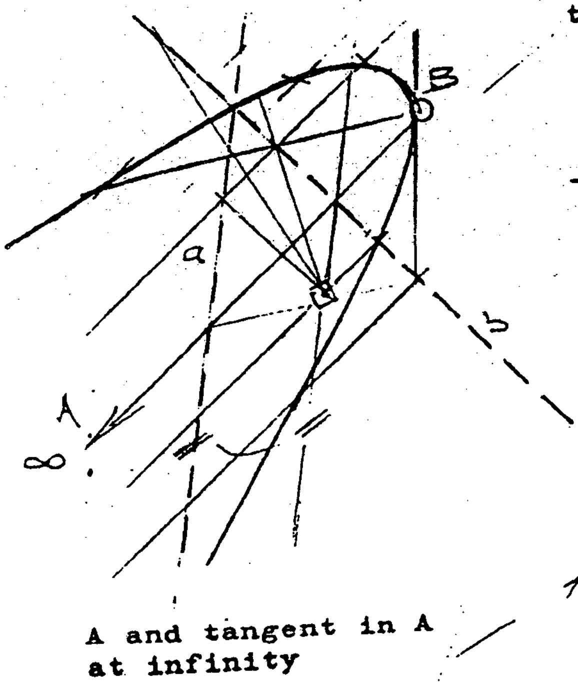


s at infinity
A on b , B on a
(a and b are asymptotes)

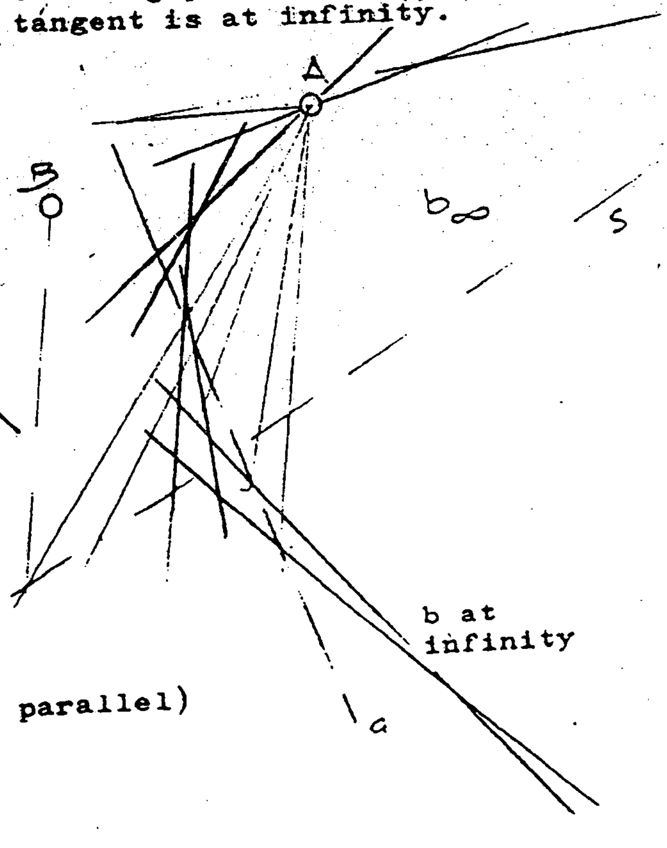


PARABOLA

A conic curve with one tangent at infinity. The tonching point of this tangent is at infinity.



A and tangent in A at infinity



b at infinity

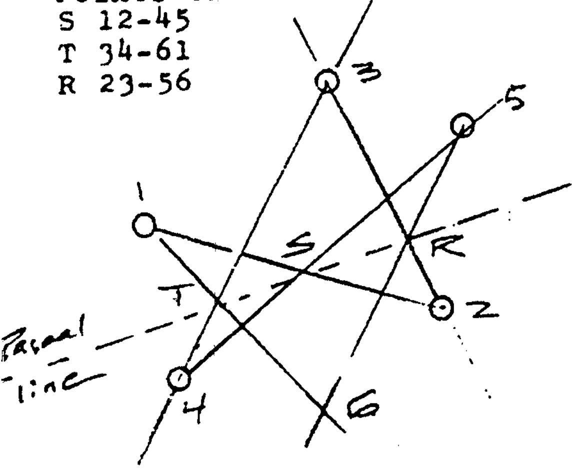
(// abbreviation for parallel)

PASCAL LINE (Blaise Pascal 1623- 1662)

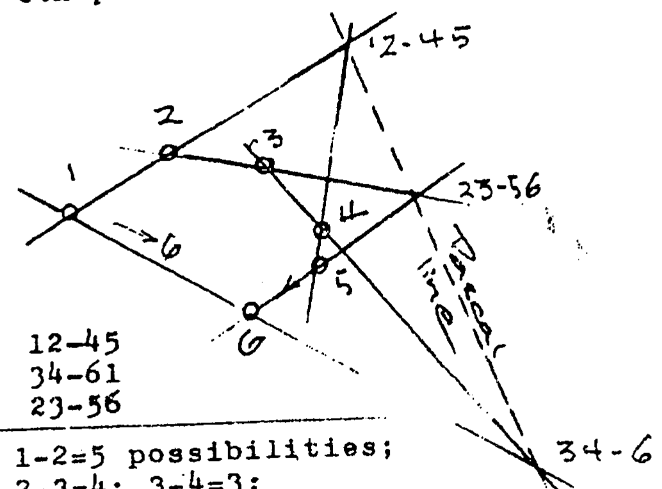
Six points on a conic curve (curve of second order) The opposite sides of this hexagon intersect on a line, the Pascal line. The intersections are collinear.

The projective construction of the pointwise conic contains the proof for this theorem.

Points of intersection :
S 12-45
T 34-61
R 23-56

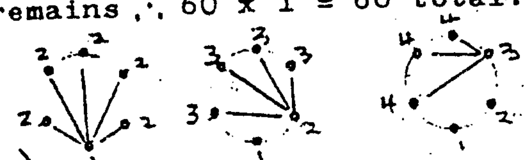


Given: 5 points. Find a 6th point on line through 1.

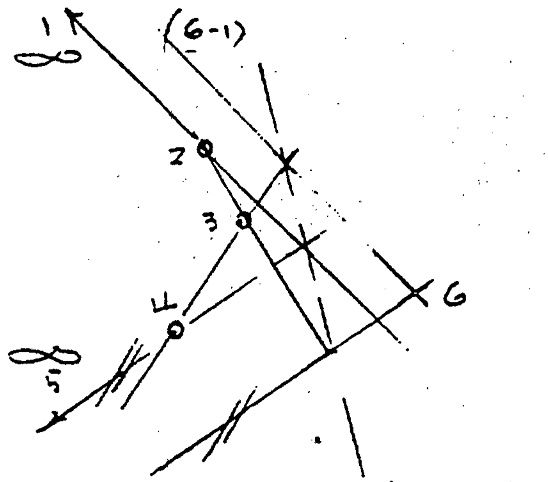


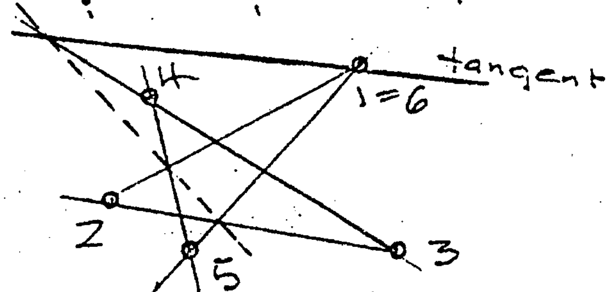
There are 60 different ways to connect 6 points with simple hexagons, i.e. 60 Pascal lines.

1-2=5 possibilities; 2-3=4; 3-4=3: multiplied = 60. One possibility remains ∴ 60 x 1 = 60 total.



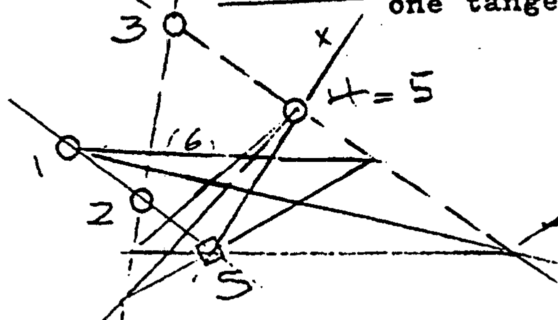
Hyperbola: point 1 and 5 at infinity.



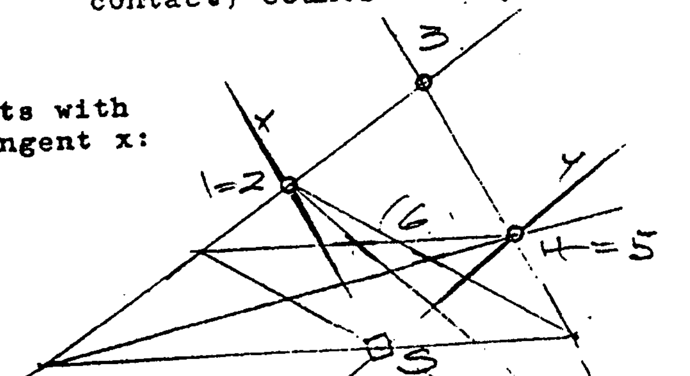


Construct the tangent- one of the points (point of contact) counts twice.

Given: 4 points with one tangent x:



find other points.



Given: 3 points with 2 tangents x,y: find other points.

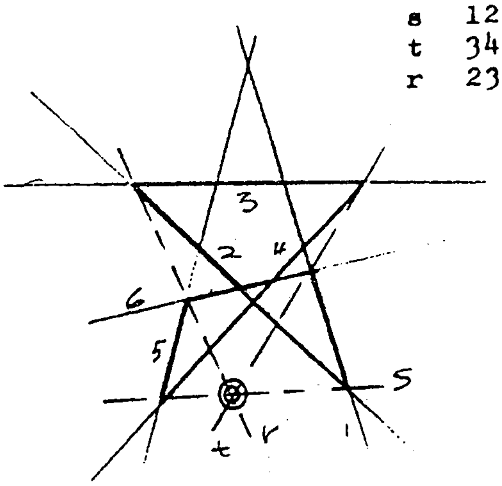
BRIANCHON POINT

(Brianchon 1785-1864)

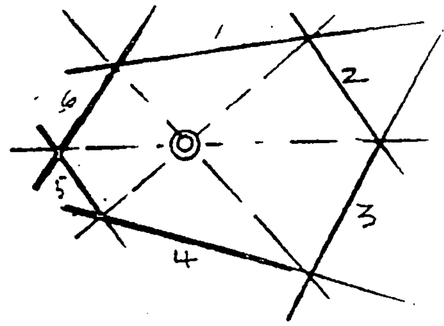
Six tangents on a conic curve.
The opposite points of this hexagram connect with lines through the Brianchon point.

The projective construction of the linewise conic shows the proof of this theorem, which is the polar opposite of Pascal's hexagon.

s 12-45
t 34-16
r 23-56

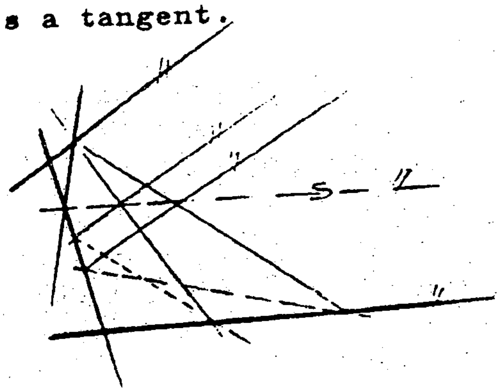


Given: five tangents.
Find a 6th tangent through any point on 1.

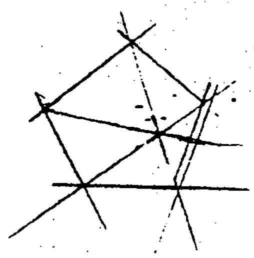


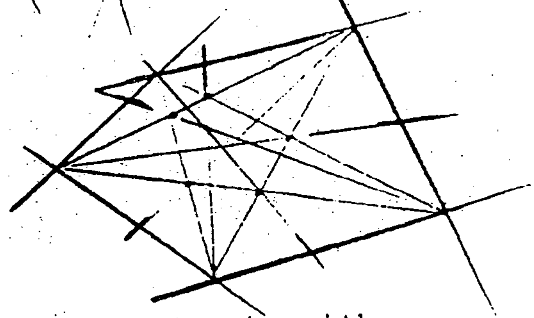
The six lines can be connected in 60 different ways, giving 60 Brianchon points.

Parabola :
the line at infinity is a tangent.

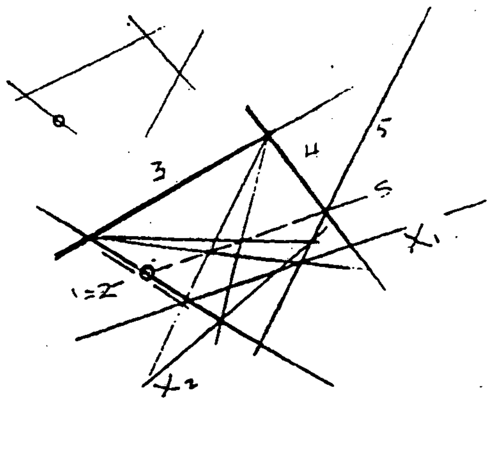


Construction of touching points on 5 tangents(one tangent counts twice).

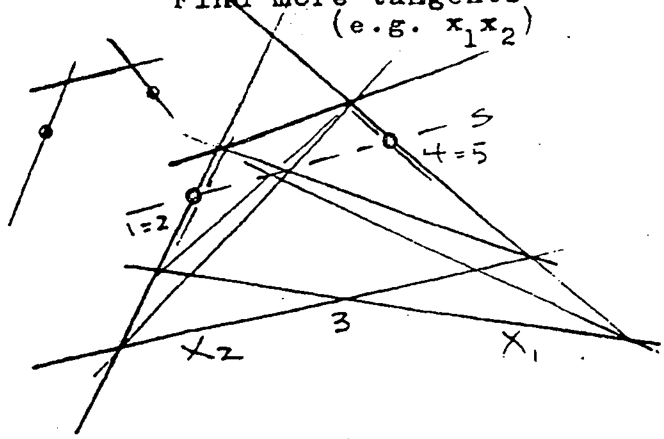




Given: 4 tangents with one touching point.
Find more tangents(e.g. x_1x_2)



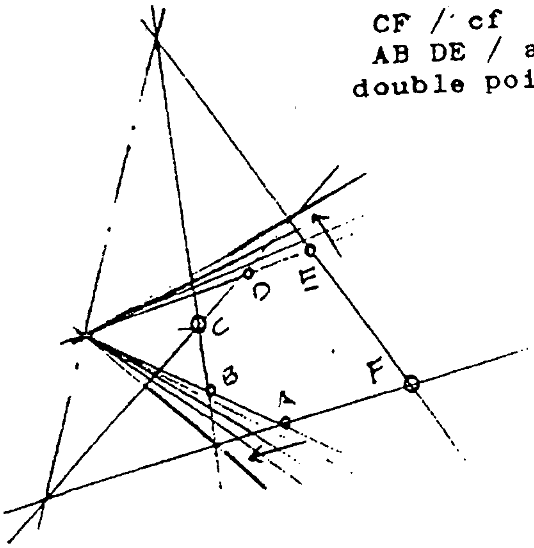
Given : 3 tangents with two touching points.
Find more tangents (e.g. x_1x_2)



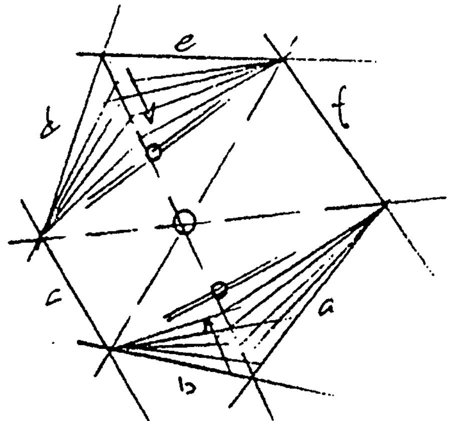
Hexagon on a conic
transformed into a
QUADRANGLE

Hexagram on a conic
transformed into a
QUADRILATERAL

CF / cf remain fixed.
AB DE / ab de move until they are
double points, double tangents.

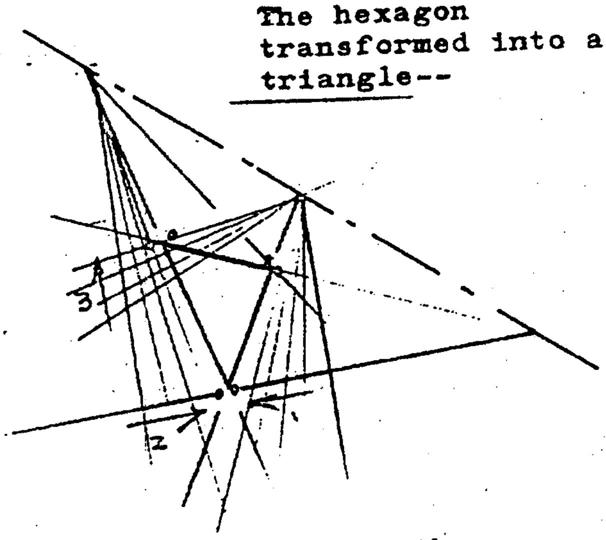


Opposite tangents intersect
on the Pascal line,
the line of intersecting
points of opp. sides.

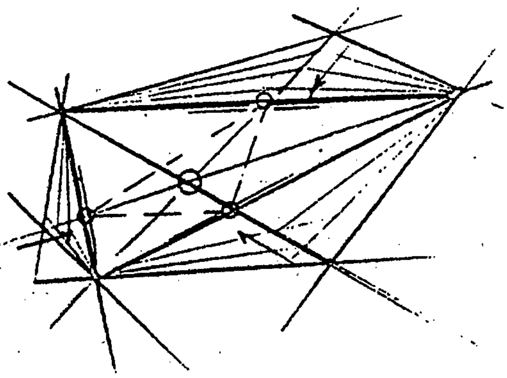


Opposite points connect
through the Brianchon point,
the point of connecting
lines of opp. points.

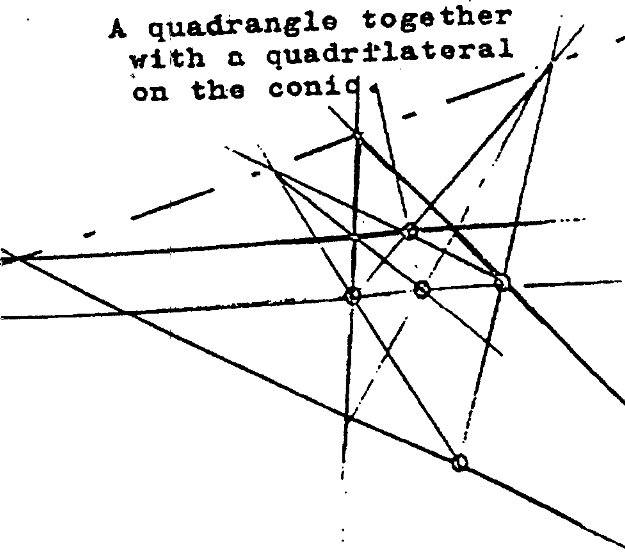
The hexagon
transformed into a
triangle--



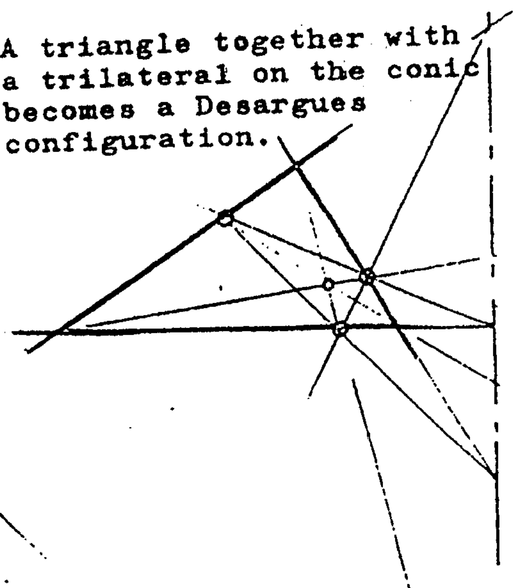
The hexagram
transformed into a
trilateral--



A quadrangle together
with a quadrilateral
on the conic.



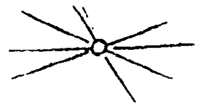
A triangle together with
a trilateral on the conic
becomes a Desargues
configuration.

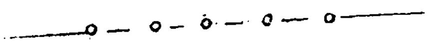


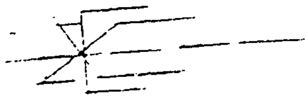
P H E N O M E N A of O R D E R

Basic configurations of the first degree :

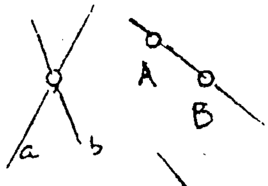
Pencil of lines | Range of points | Pencil of planes



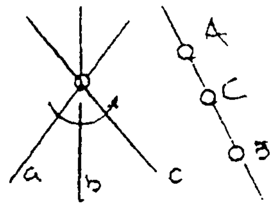




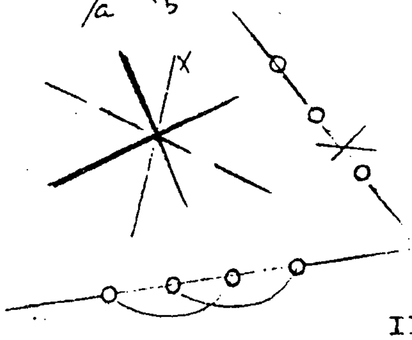
There follow some self-evident facts. To grasp them conceptually is both necessary and fruitful.



The sense of direction is not determined.



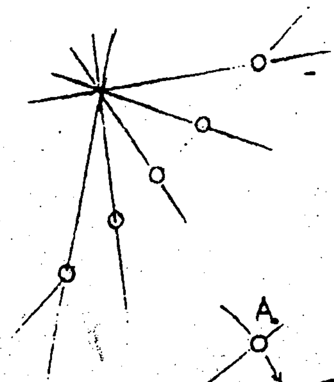
Only one sense of direction is possible.



I. With three given elements, another element will always be separated from any one of the three by the other two.

Four elements always have two pairs separating each other.

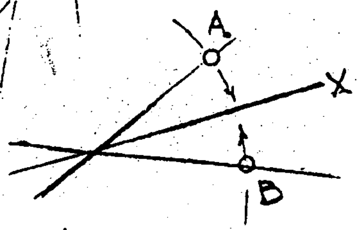
II. A number of elements is by nature ordered into a particular sequence or cycle.



III. By intersecting or connecting, the order is maintained.

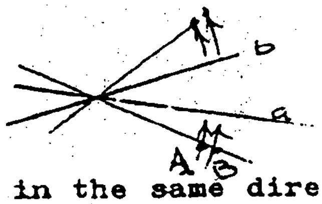
"Between" is complemented by "separating" in projective geometry.

IV. Continuity: if two lines move along an angle field of a pencil at the same time -



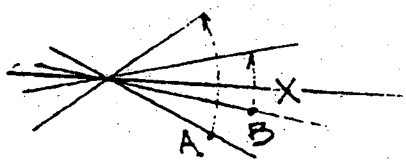
in the opposite direction :

they meet in one line (x).



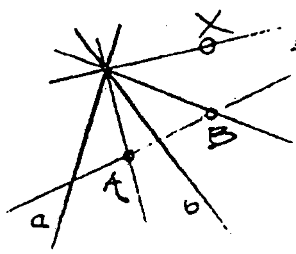
in the same direction: (starting and finishing together)

they are separated between a and b.



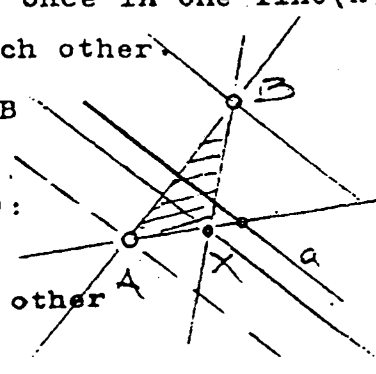
in the same direction :

they meet at least once in one line (x).

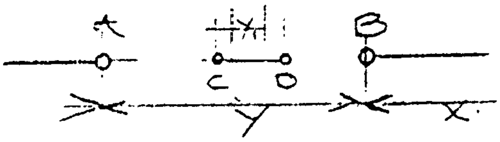


A, B and a, b separate each other. Any third point X is then separated from either A or B by the two lines.

——— Special case (right): b at infinity. The line a intersects the inner segment of AB and only one other inner segment of the sides of the triangle ABX .



Between two pairs of elements, which are not separating each other, there will always be at least one third pair which separates both other pairs harmoniously.

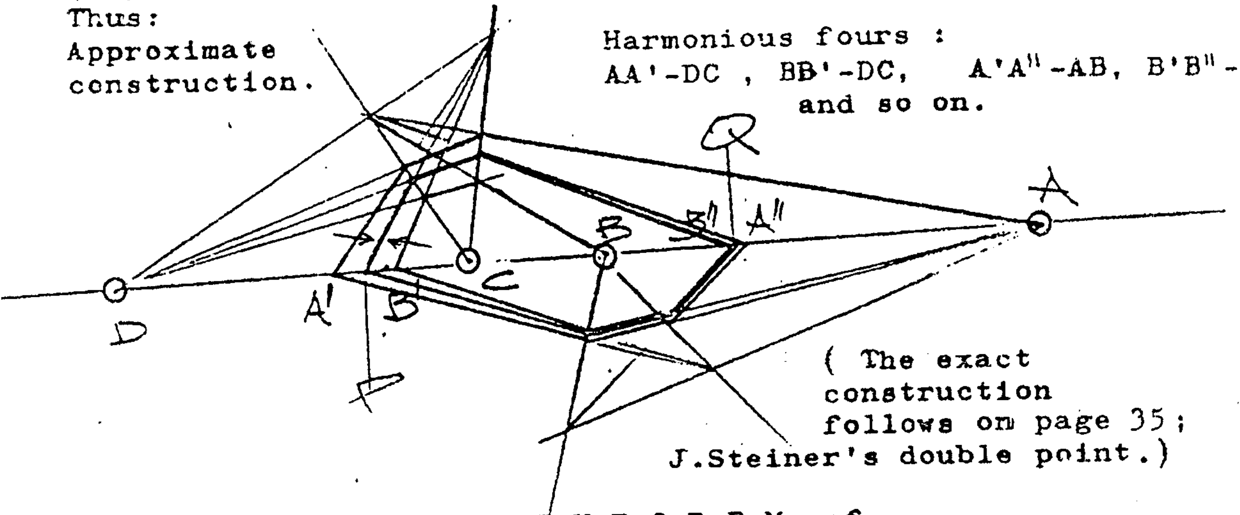


Y is any harmonious fourth point with respect to a point X between A and B. Y_1 is any harmonious fourth point with respect to a point X between C and D.

According to item IV example 3 (page 25) Y and Y_1 will meet in a point (say Q). This point with its corresponding point (say P) will separate the pairs AB and CD harmoniously.

Thus:
Approximate construction.

Harmonious fours : AA'-DC , BB'-DC, A'A''-AB, B'B''-AB . and so on.

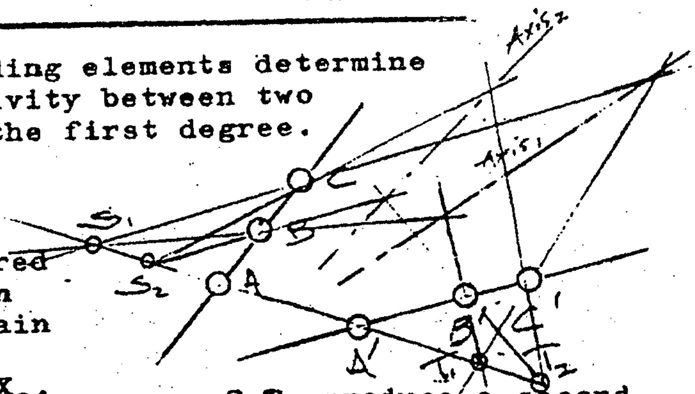


(The exact construction follows on page 35; J.Steiner's double point.)

F U N D A M E N T A L T H E O R E M of Projective Geometry

Three pairs of corresponding elements determine one and only one projectivity between two basic configurations of the first degree.

ABC $\barwedge$ A'B'C'

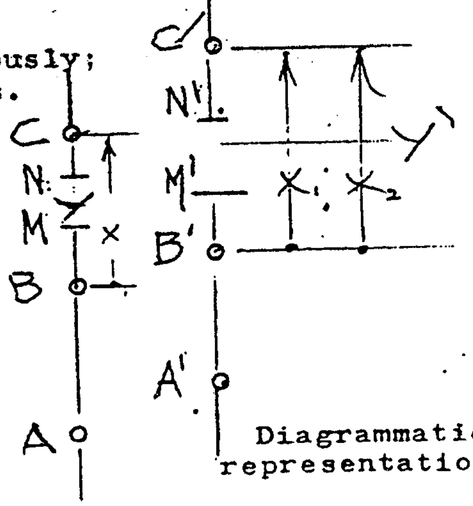


PROOF : Two chains of construction are considered from S_1T_1 and S_2T_2 chosen anywhere on A-A'. One chain translates from X to X_1 and the other from X to X_2. Now assume that the movements of X_1 and X_2 do not coincide from M' to N'. However: YA - MN separate each other harmoniously; Y'A' - M'N' are then also harmonious.

The assumption that the two chains separate from M' to N' is contradicted, for harmonious fours are maintained by projection. X_1 and X_2 remain together in their movements.

With this the one-to-one correspondence is established.

S_2T_2 produce a second axis for a second chain.



Diagrammatic representation

The following drawing on the next page should be studied to make this proof understandable.

Fundamental Theorem of Projective Geometry

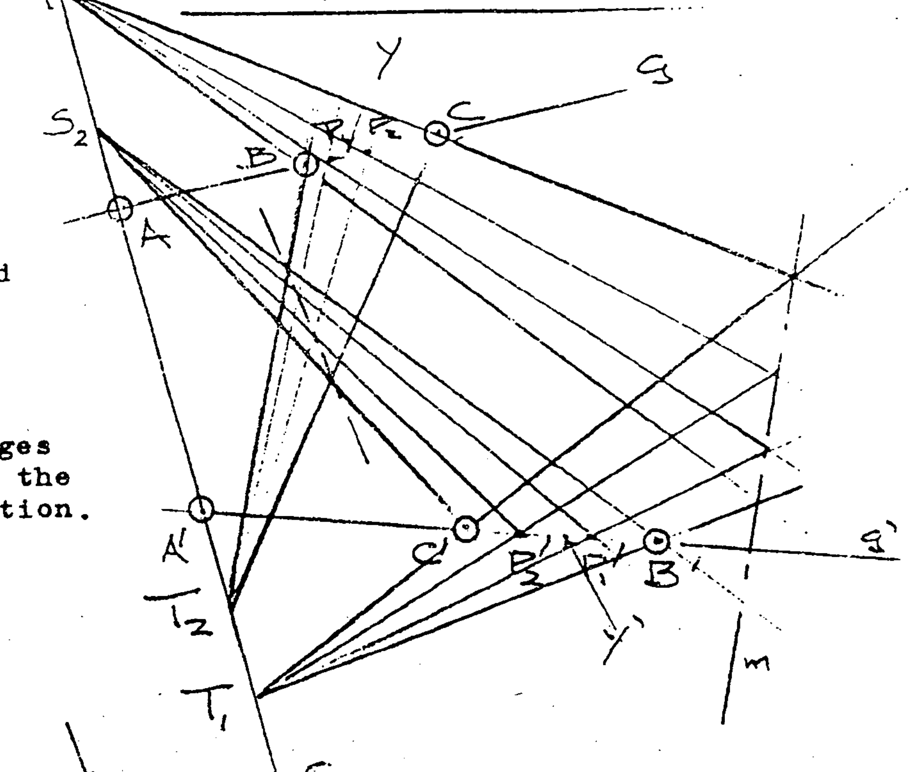
Two projective ranges:
ABC ⊼ A'B'C'

Construction with two chains S_1T_1 - S_2T_2

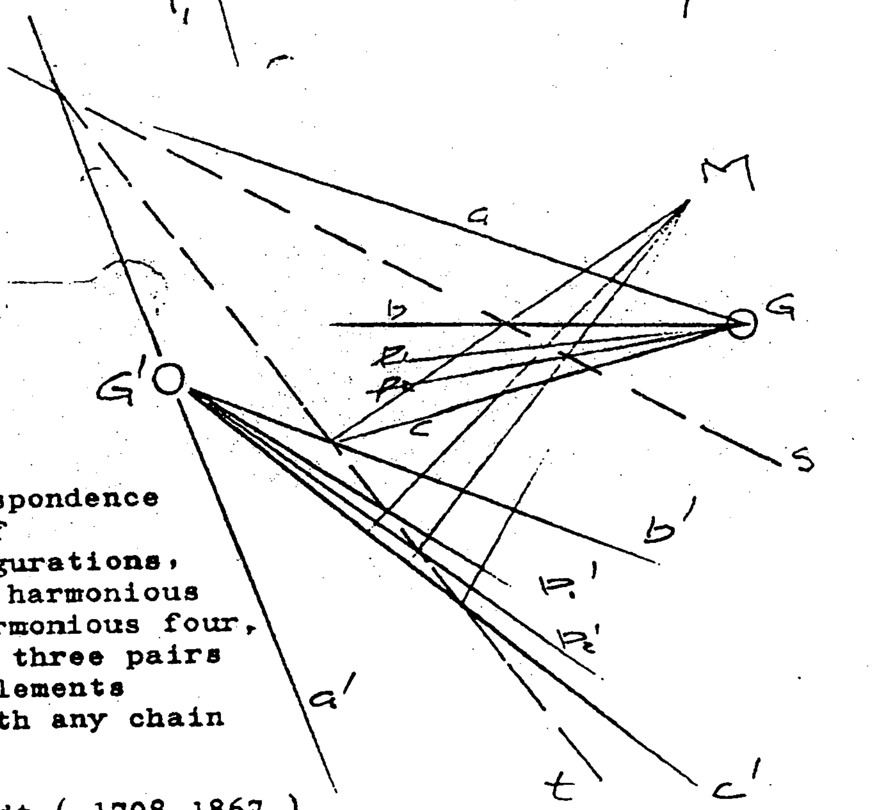
Y harm. separated from A by P_1P_2

Y' harm. separated from A' by $P_1'P_2'$

The ratio of the two speeds of the movements of the corresponding ranges is independent of the chain of construction.



Two projective pencils :
abc ⊼ a'b'c'
and so on,
polar to the above.



A reversible correspondence between elements of first degree configurations, which refers every harmonious four to another harmonious four, is determined with three pairs of corresponding elements and is realised with any chain of construction.

Christian von Staudt (1798-1867) has explained projectivity in this manner.

P H E N O M E N A of O R D E R (continued)

Characteristics of the three basic configurations of the first degree : Range of points, pencil of lines and pencil of planes.

All three form sets which are o r d e r e d , c y c l i c and d e n s e ("dicht").

DENSITY : Between any two elements of an ordered set is at least one further element of the set.

These basic configurations are not only dense but also C O N T I N U O U S ("stetig")
This means that division of a section into two ordered groups is achieved by an element of that section.

A continuous set is without a gap.
A finite set can only be divided by way of its gaps.

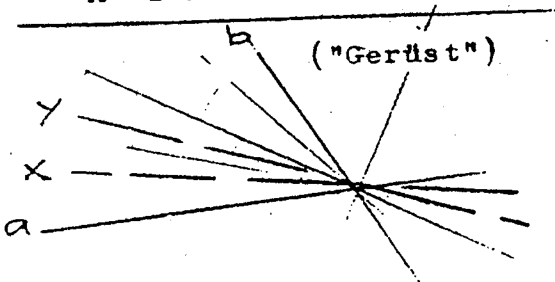
C O U N T A B L E Set : The elements can be numbered; e.g. constructing from three points on a line an endless number of further points. This produces a countable yet infinite set of points.

An ordered set of elements which is continuous can not be c o u n t a b l e ("abzählbar").

There is the C O U N T A B L E I N F I N I T Y
and the N O N - C O U N T A B L E I N F I N I T Y
(" das überabzählbar Unendliche")

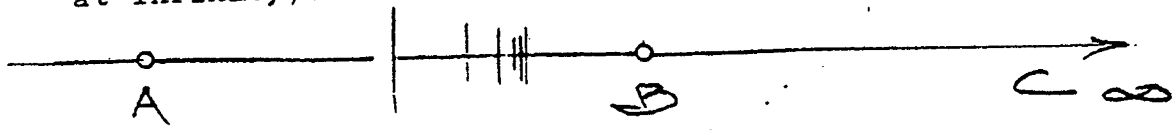
Not all positions of a sequence of points can be brought to awareness in our ordinary consciousness. On a straight line (as a continuous sequence) there does not exist a neighbour point to a point.

A S K E L E T O N



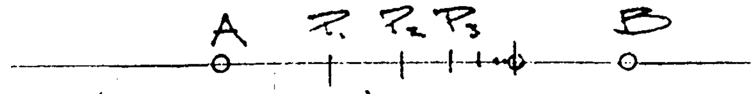
for example, of a pencil of lines. From two given lines (a and b) divide the interval continuously in two. Let G be this countable infinite set of dividing lines, with x,y as any interval and not necessarily belonging to G. This interval certainly always contains elements of G. G is called a skeleton.

With respect to three fixed points on a line (one point at infinity) a continuous harmonious division is possible.



C L U S T E R P O I N T or LIMIT POINT ("HAUFUNGSPUNKT")

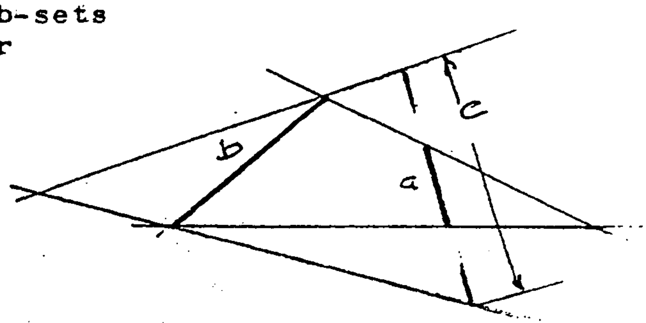
If in a sequence of points two points A and B exist, so placed that P_2 lies, with regard to A, inside P_1B; P_3, with regard to A, inside P_2B; ... P_n, with regard to A, inside $P_{n-1}B$, the sequence of points is said to be monotonous. A monotonous sequence of points determines one c l u s t e r p o i n t.

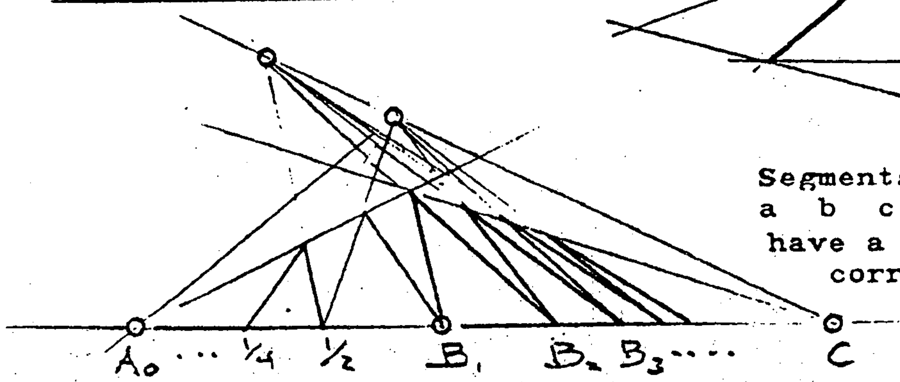


P O W E R : ("Mächtigkeit")

Two sets have the same power if they are in a definite one-to-one correspondence, which is also reversible.

A definite set contains sub-sets which are of the same power as the complete set. This is impossible with a finite set of elements.





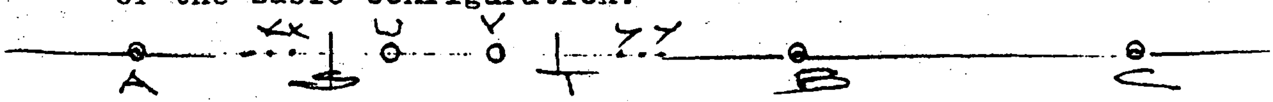
Segments a b c are perspective, have a one-to-one correspondence and therefore are of the same power.

R A T I O N A L D I V I S I O N o12⊼123⊼234,...

A net with respect to three points A, B and C (above left).

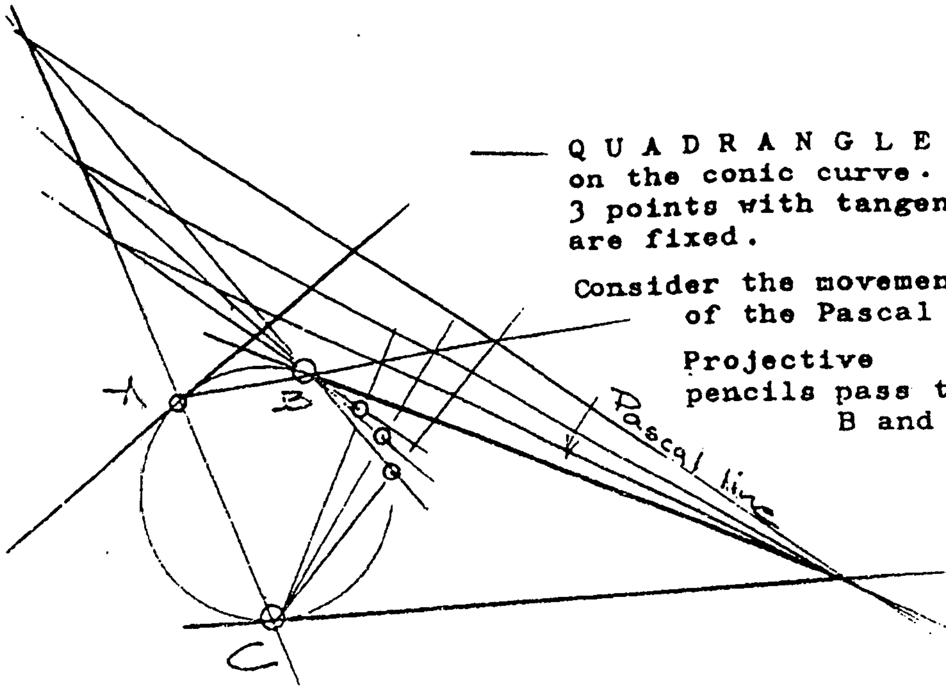
E v e r y i n t e r v a l contains points of the net.

This net, which is determined by three elements of a basic configuration, is a skeleton of the basic configuration.



U and V shall determine any interval but not be separated by any two of the three given points A B C . A U V B C are in natural order.

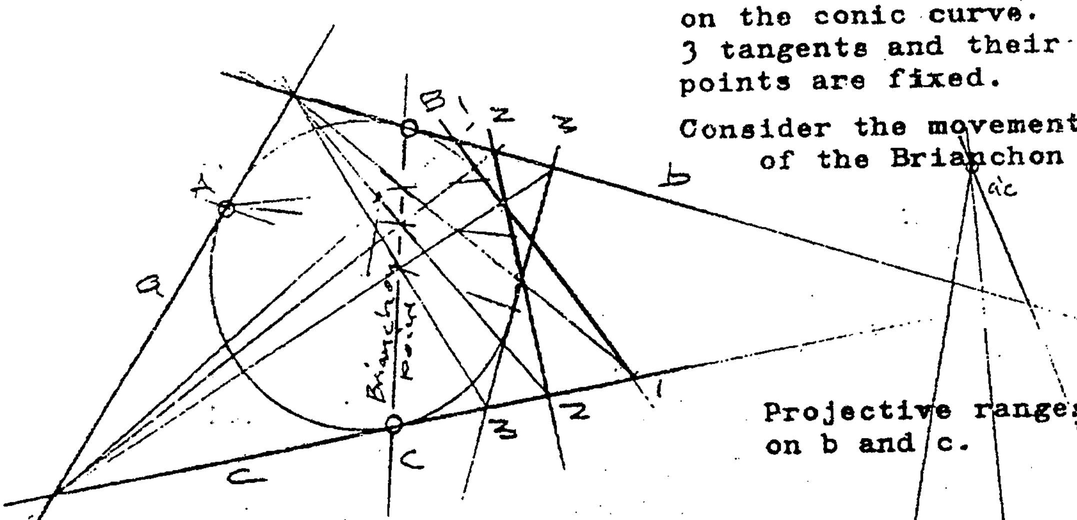
Between U and V with regard to C are points of the net. If U and V are points of the net, the mirror of C with regard to U and V produces a point of the net(harmonious 4th point). If U and V are not points of the net, then from U to A one meets either a point X of the net or a cluster point S of the net. Correspondingly from V to B one meets the points Y or T. We can choose points of the net within any proximity to X,Y,S or T and the harmonious 4th point with regard to C produces a net point between U and V. Thus indeed every interval contains points of the net.



QUADRANGLE
on the conic curve.
3 points with tangents are fixed.

Consider the movement of the Pascal line.

Projective pencils pass through B and C.



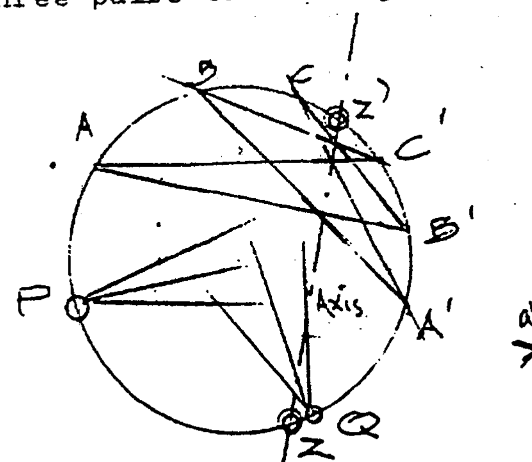
QUADRILATERAL
on the conic curve.
3 tangents and their touching points are fixed.

Consider the movement of the Brianchon point.

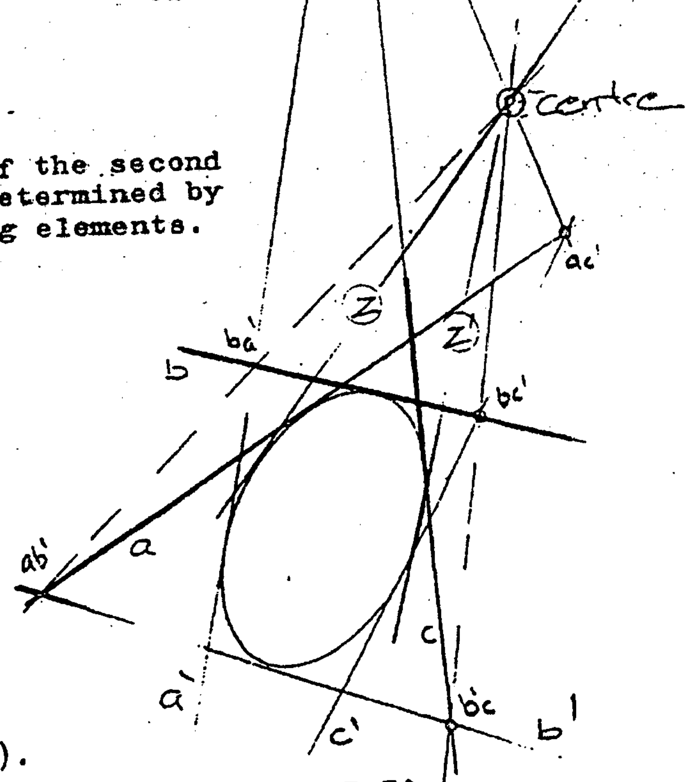
Projective ranges lie on b and c.

PROJECTIVITY IN ITSELF

On a range or on a pencil of the second degree: a projectivity is determined by three pairs of corresponding elements.



Projective pencils from 2 points of the conic (PQ). Cross lines produce the axis.

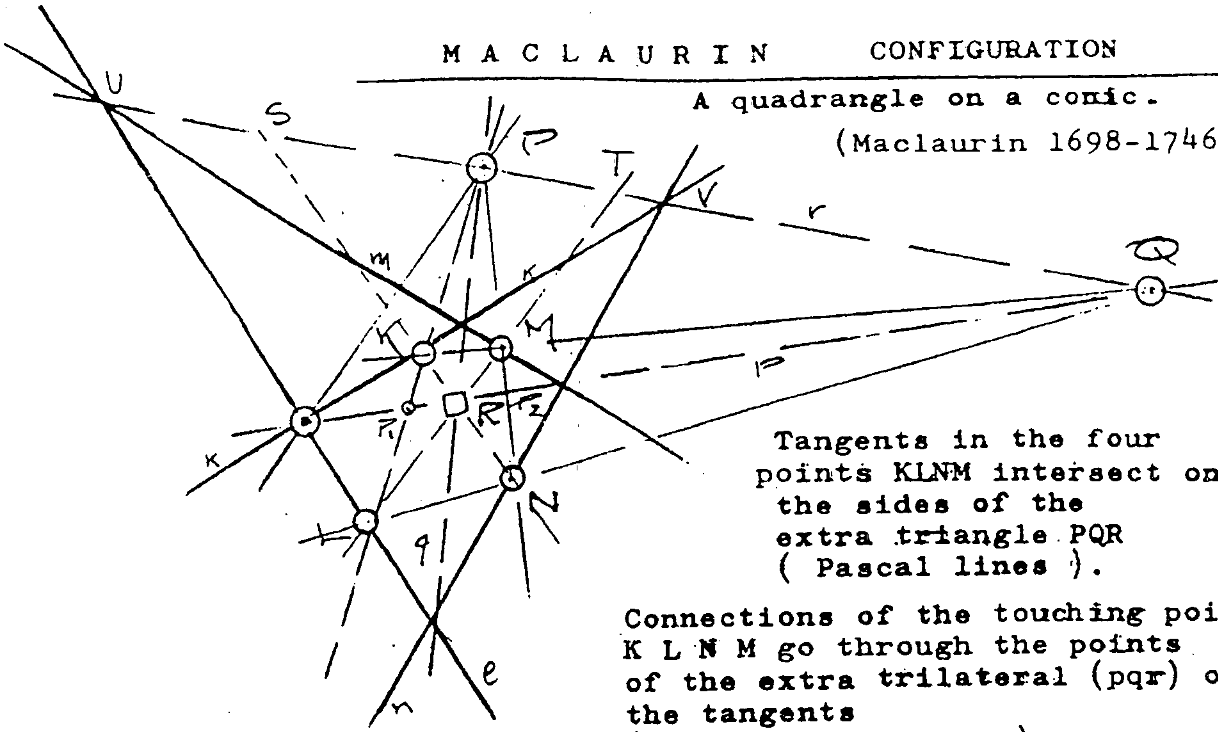


double points Z Z'
double tangents z z'

M A C L A U R I N CONFIGURATION

A quadrangle on a conic.

(Maclaurin 1698-1746)



Tangents in the four points KLNM intersect on the sides of the extra triangle PQR (Pascal lines).

Connections of the touching points K L N M go through the points of the extra trilateral (pqr) of the tangents (Brianchon points).

The extra trilateral of the tangents(pqr) coincides with the extra triangle of the touching points(PQR) - but RST and RUV do not coincide as in thebasic harmonious configuration.

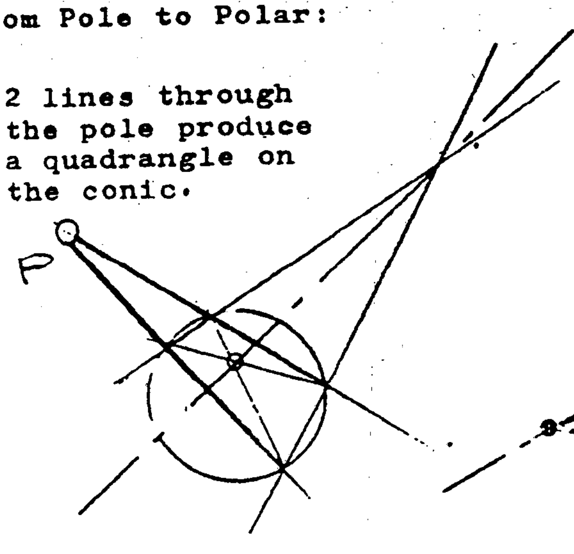
PP_1 - KL are harmonious four points (look at RMQN). There are also harmonious four lines in the intersection of kl. Any other line through P therefore produces the same line p (e.g. by means of alternative points M, N).
To every point P corresponds, therefore a definite line p with respect to the conic curve. They are called the P O L E and the P O L A R L I N E.

If the Maclaurin configuration becomes the Fundamental Harmonious Configuration the four points and the four tangents are harmonious points/tangents with respect to the conic.

Construction of pole and polar line —

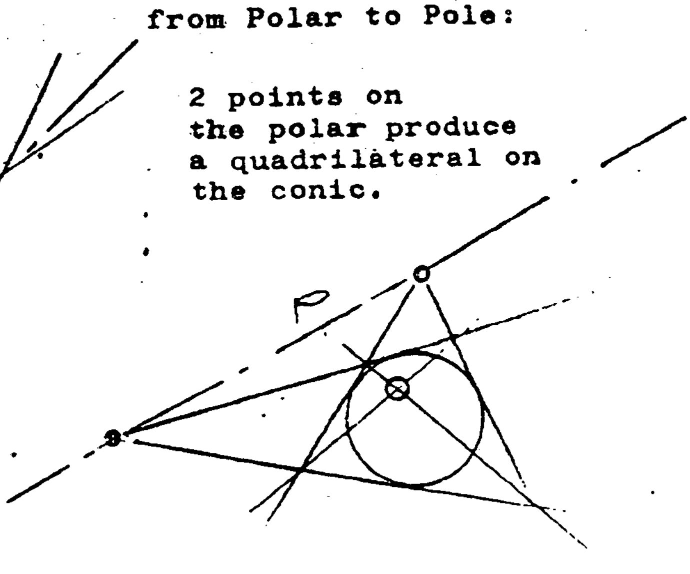
from Pole to Polar:

2 lines through the pole produce a quadrangle on the conic.



from Polar to Pole:

2 points on the polar produce a quadrilateral on the conic.



P O L E and P O L A R on the conic curve

The extra triangle of the quadrangle on the conic
or the extra trilateral of the quadrilateral (four tangents)
on the conic constitute a

P O L A R T R I A N G L E OF THE CONIC

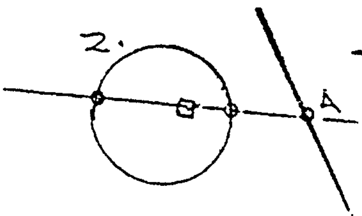
The sides are the polars of the opposite corners.

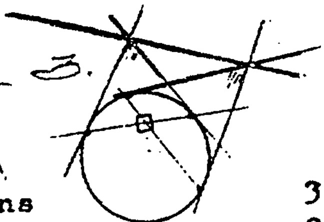
The corners are the poles of the opposite sides.

The polar triangle is self-polar.

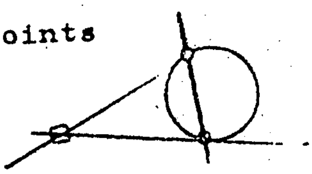
The P O L A R CONTAINS

1. Two extra points of every quadrangle for which the pole is an extra point.
2. Points (A) harm. separated by the curve from the pole.



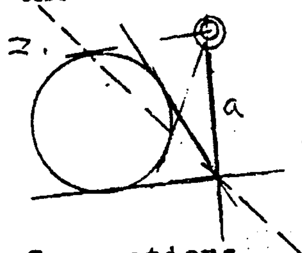


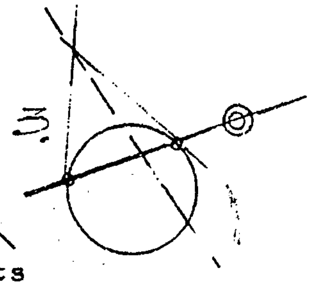
3. Intersections of tangents whose touching points connect through the pole.
4. Touching points of tangents through the pole.



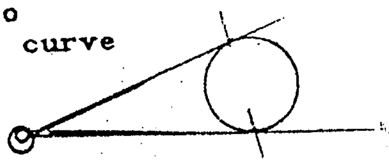
The P O L E CONTAINS

1. Two extra sides of every quadrilateral for which the polar is an extra side.
2. Lines (a) harm. separated by the curve from the polar.





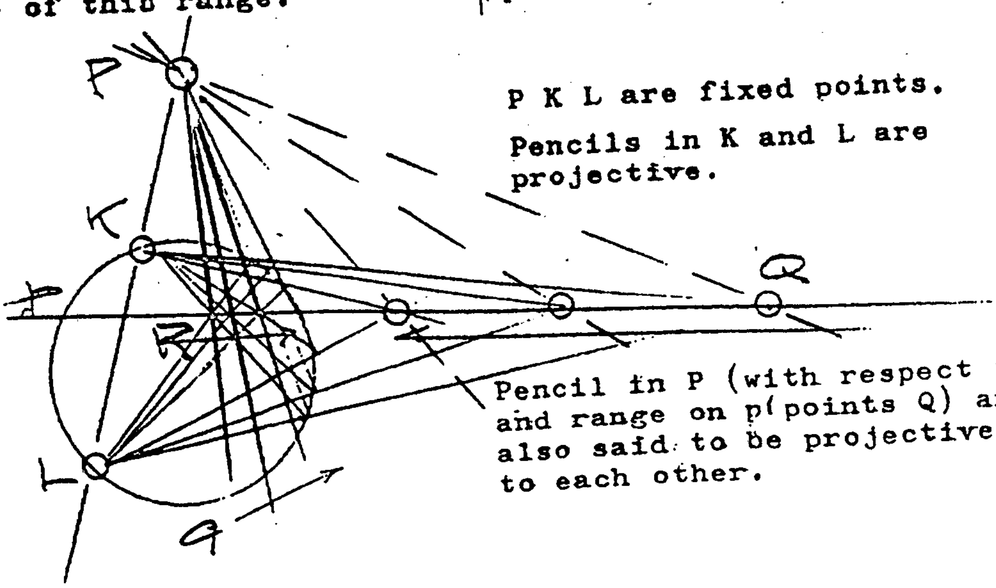
3. Connections of touching points whose tangents intersect on the polar.
4. Tangents to points on the curve on the polar.



The FUNDAMENTAL THEOREM of the THEORY of POLE and POLAR

All the polars to points of a range go through the pole of this range.

All the poles to lines of a pencil are on the polar of this pencil.

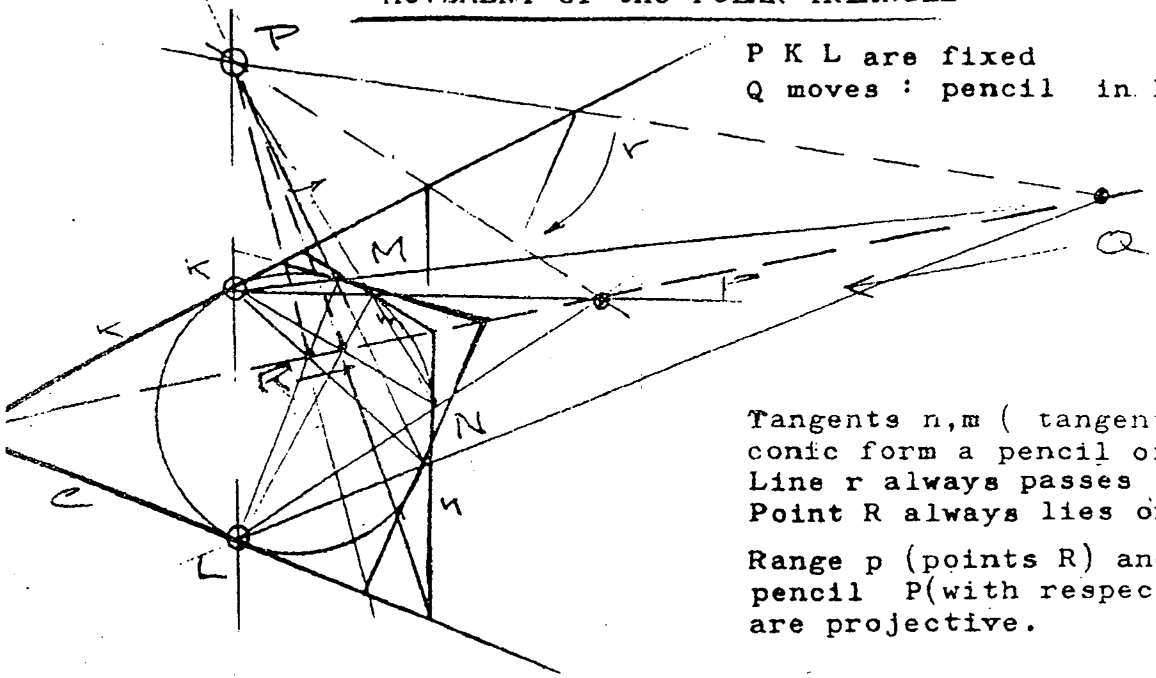


P K L are fixed points.

Pencils in K and L are projective.

Pencil in P (with respect to R) and range on p (points Q) are also said to be projective to each other.

MOVEMENT of the POLAR TRIANGLE



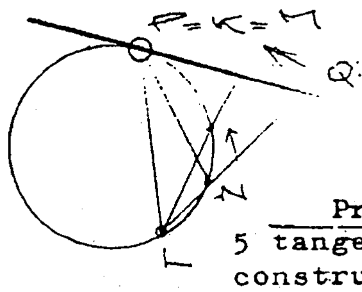
P K L are fixed
Q moves : pencil in P

Tangents n,m (tangents to a conic form a pencil of 2nd degree)
Line r always passes through P.
Point R always lies on p.

Range p (points R) and pencil P(with respect to Q) are projective.

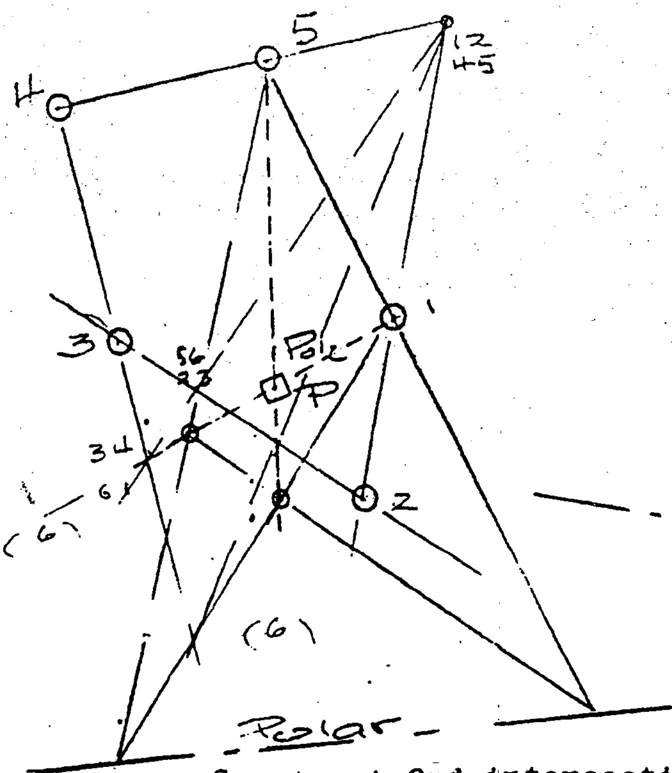
If pole P is on the conic, the tangent is the polar line.

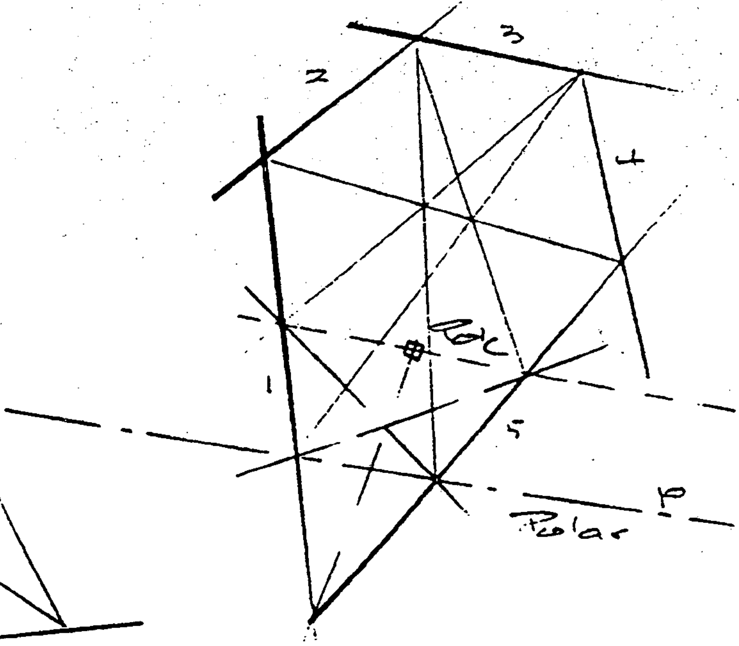
P K M become identical.



Problem: given 5 points and pole (P) construct the polar line------

Problem: given 5 tangents and polar(p.) construct the pole-----



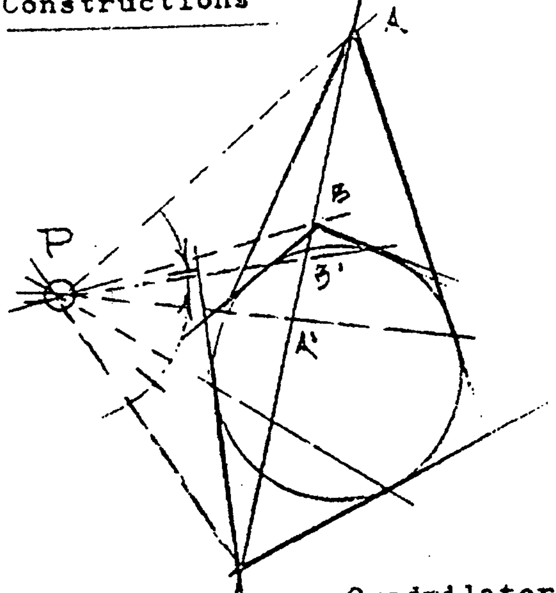


Construct 2nd intersections with the conic on lines 1P and 5P. Quadrangle + extra triangle produces the polar line.

Construct 2nd tangents through 1p and 5p. Quadrilateral + extra trilateral produces the pole.

MOVEMENT of the POLAR TRIANGLE

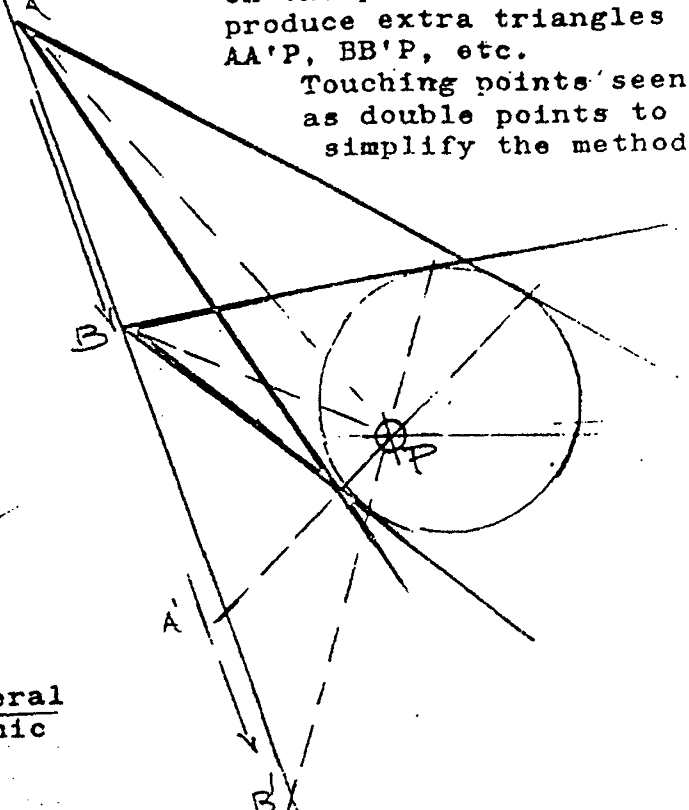
Constructions



Quadrilateral on the conic

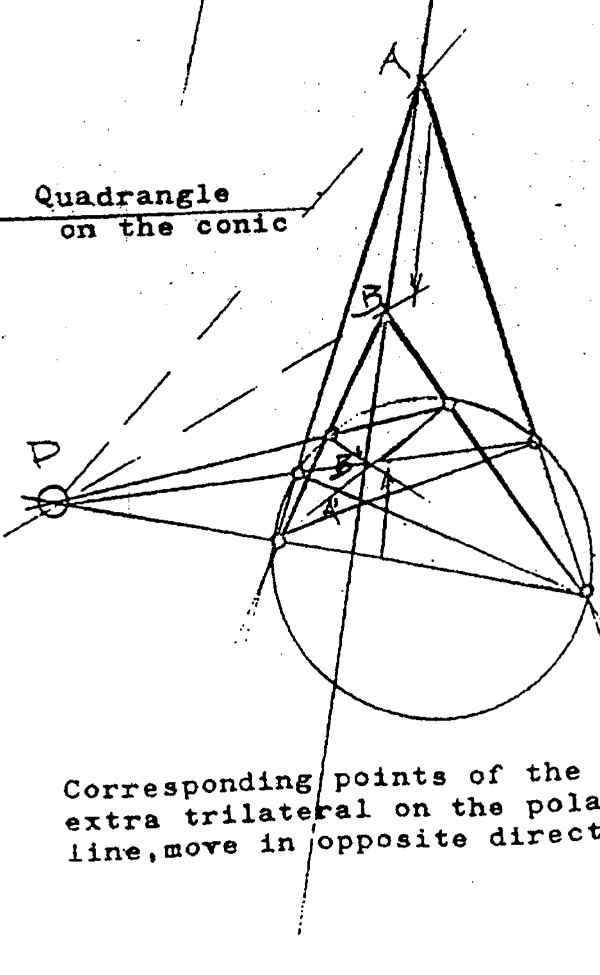
pole - outside (pointwise)
polar - outside (line wise)

Tangents from points on the polar line--- produce extra triangles AA'P, BB'P, etc.
Touching points seen as double points to simplify the method.



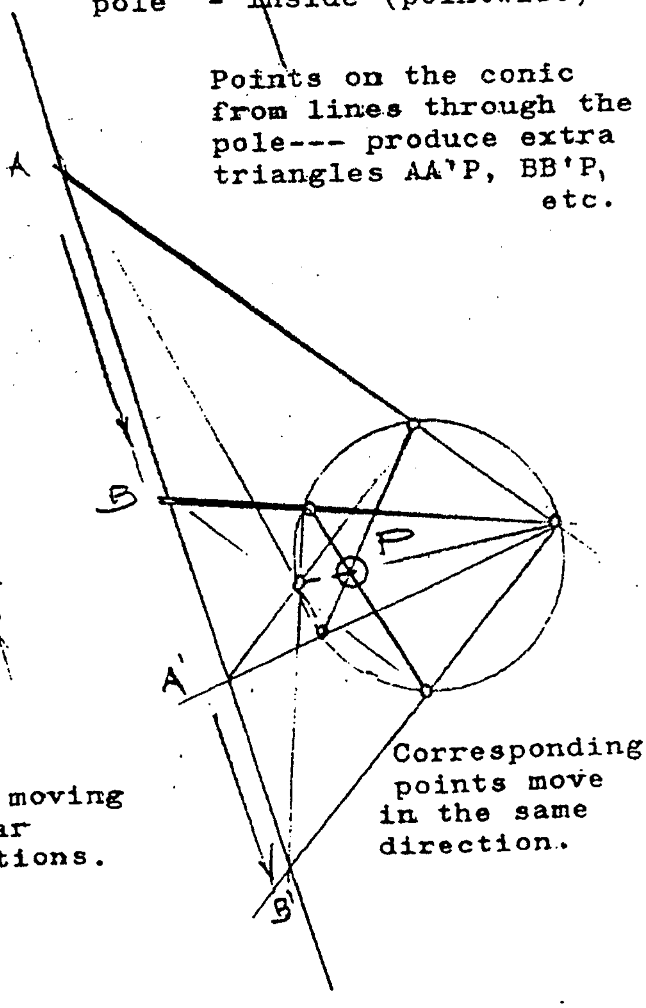
polar - inside (linewise)
pole - inside (pointwise)

Quadrangle on the conic



Corresponding points of the moving extra trilateral on the polar line, move in opposite directions.

Points on the conic from lines through the pole--- produce extra triangles AA'P, BB'P, etc.

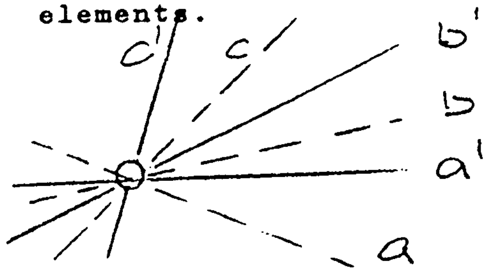


Corresponding points move in the same direction.

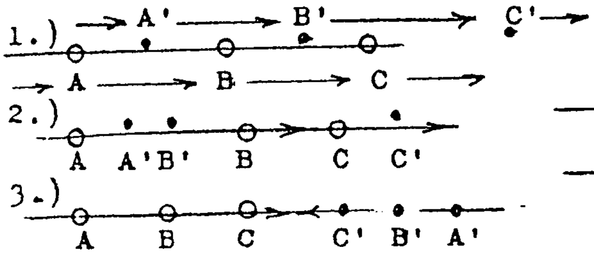
P R O J E C T I V I T Y in I T S E L F

Two pencils through the same point and
two ranges on the same line -
have the same base .

A projectivity between two such basic configurations with a common base is called a projectivity in itself and is determined by three corresponding pairs of elements.



Cyclic ordering :
If corresponding elements (e.g.A,A') fall together we have double points or double lines. A projectivity can have no more than two double elements otherwise it is "identity".

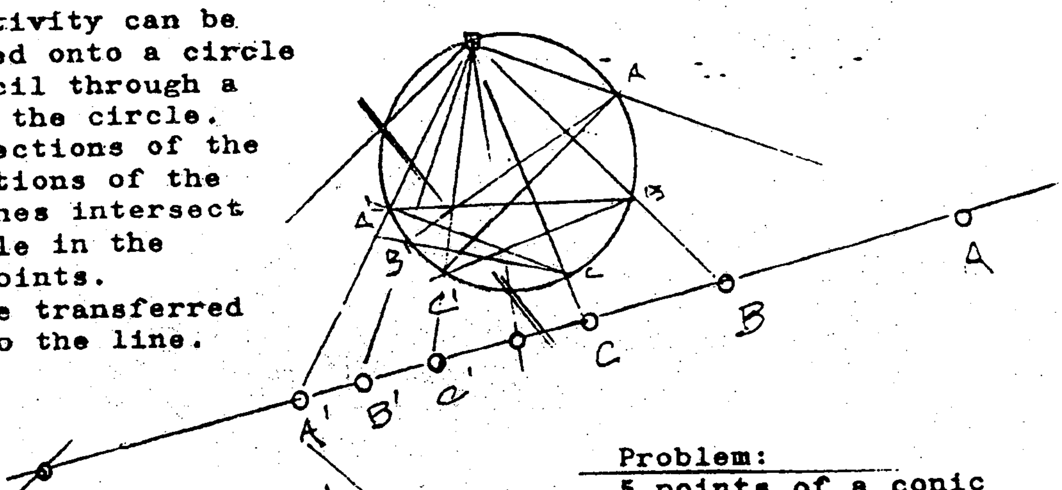


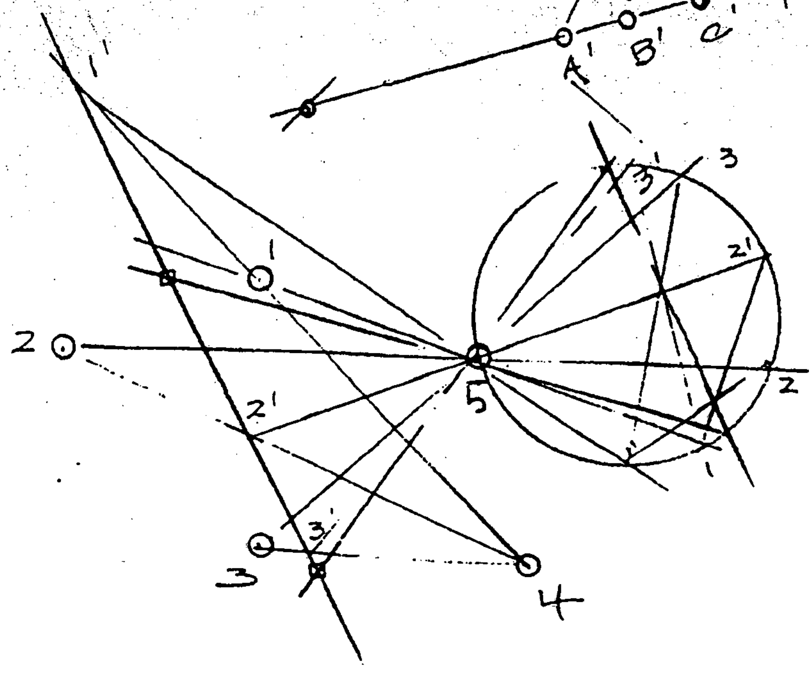
If the movement of two cycles (case 1)is in the same direction, there is no double element;
two cycles (case 2)move in the same direction- two double elements;
two cycles (case 3)move in opposite directions- two double elements.

Points move: A to B to C to A and A'to B'to C'to A'.

DOUBLE-POINT CONSTRUCTION (Jakob Steiner 1796-1863)

A projectivity can be tranferred onto a circle by a pencil through a point on the circle. The connections of the intersections of the cross lines intersect the circle in the double points. These are transferred back onto the line.





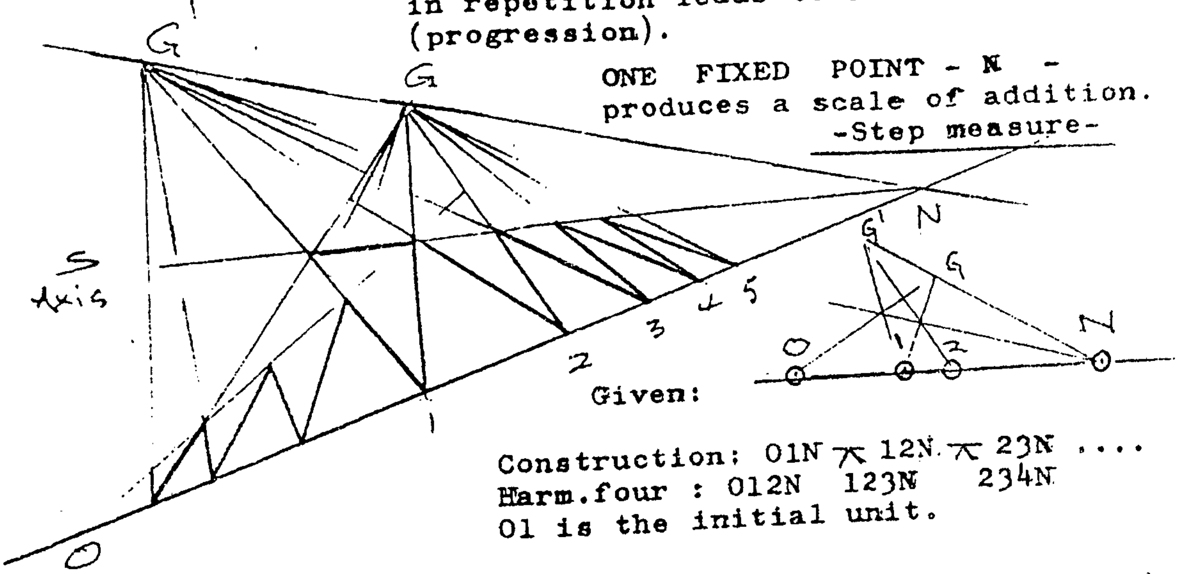
Problem:
5 points of a conic are given.
Construct intersections of a line with the conic.
(Projective pencils from 4 and 5 are transferred to the conic.
Double-point construction).

Solve the polar problem :
connection of a point with a conic - - -

S C A L E of A D D I T I O N

Projectivity into itself in repetition leads to a scale (progression).

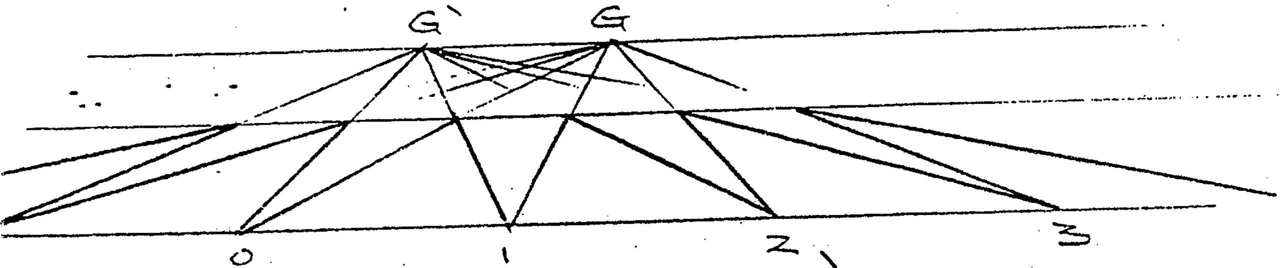
ONE FIXED POINT - N - produces a scale of addition.
-Step measure-



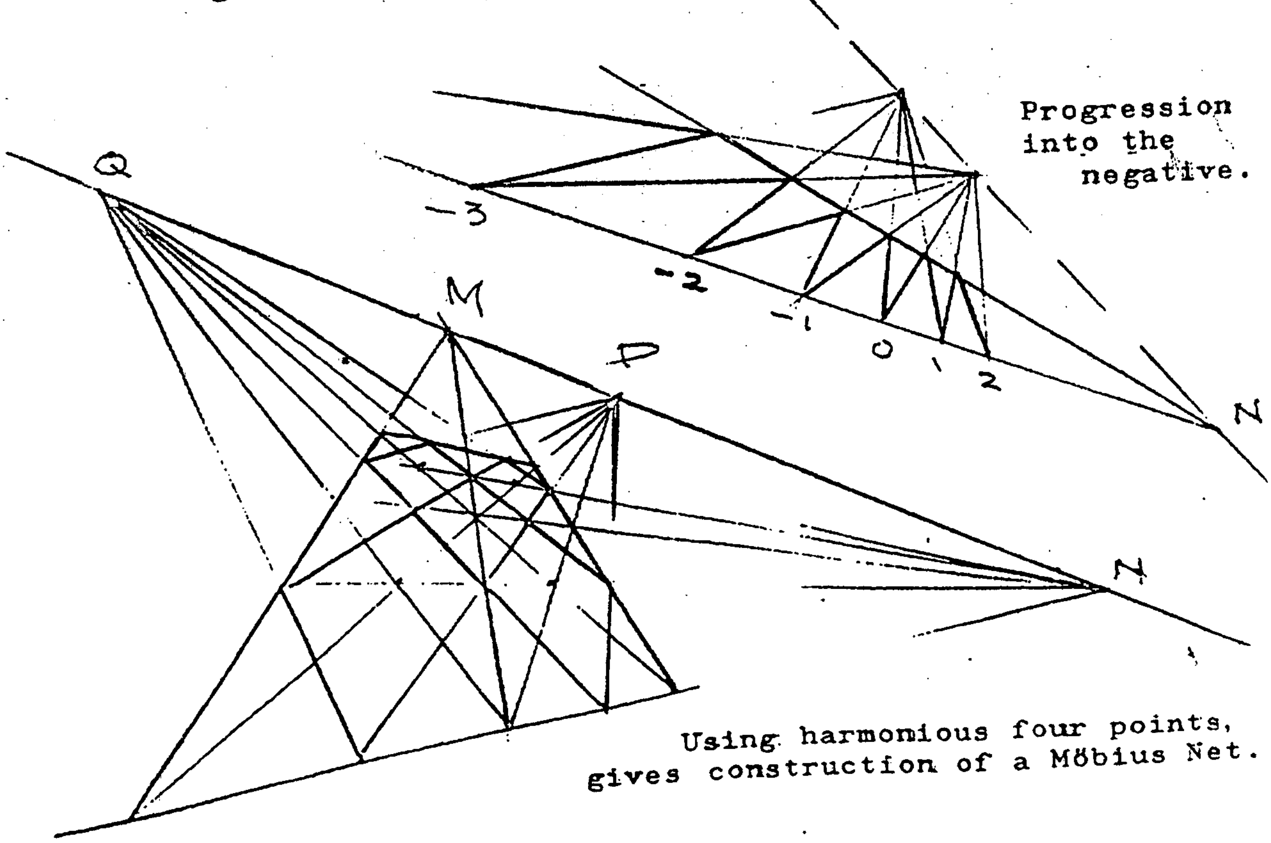
Given:

Construction: 01N ⊼ 12N ⊼ 23N ....
Harm.four : 012N 123N 234N
01 is the initial unit.

Fixed point N at infinity produces arithmetically equal distances.

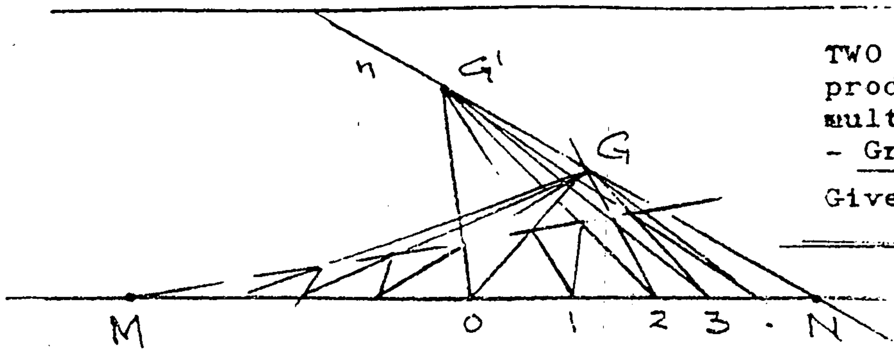


Progression into the negative.



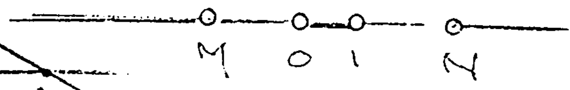
Using harmonious four points, gives construction of a Möbius Net.

SCALE of MULTIPLICATION



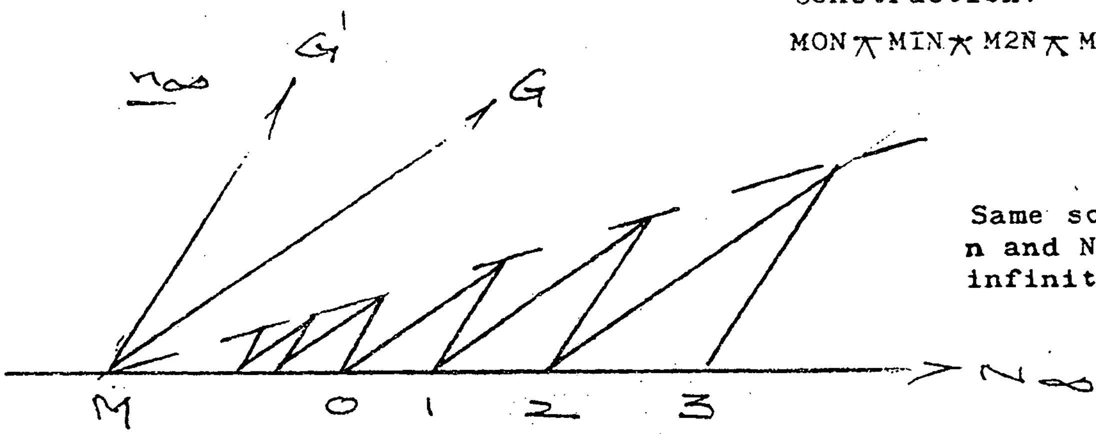
TWO FIXED POINTS -N M- produce a scale of multiplication.
- Growth measure -

Given:



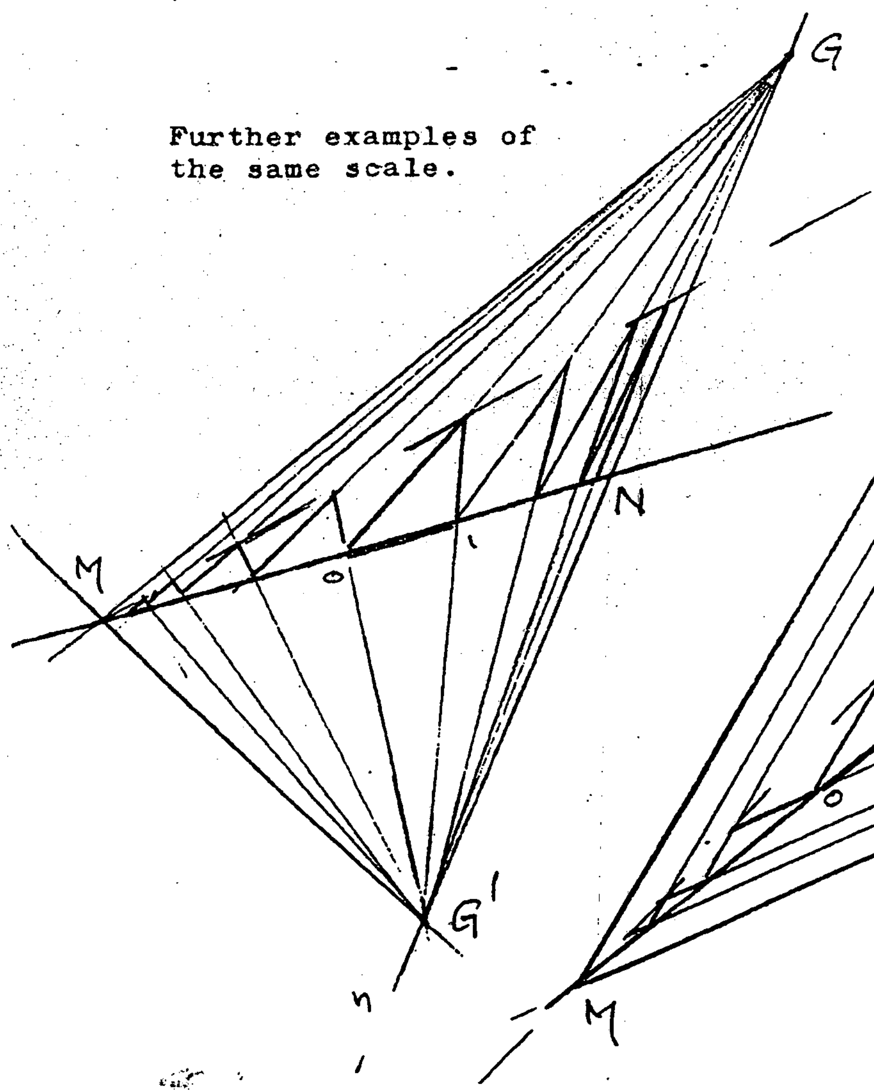
Construction:

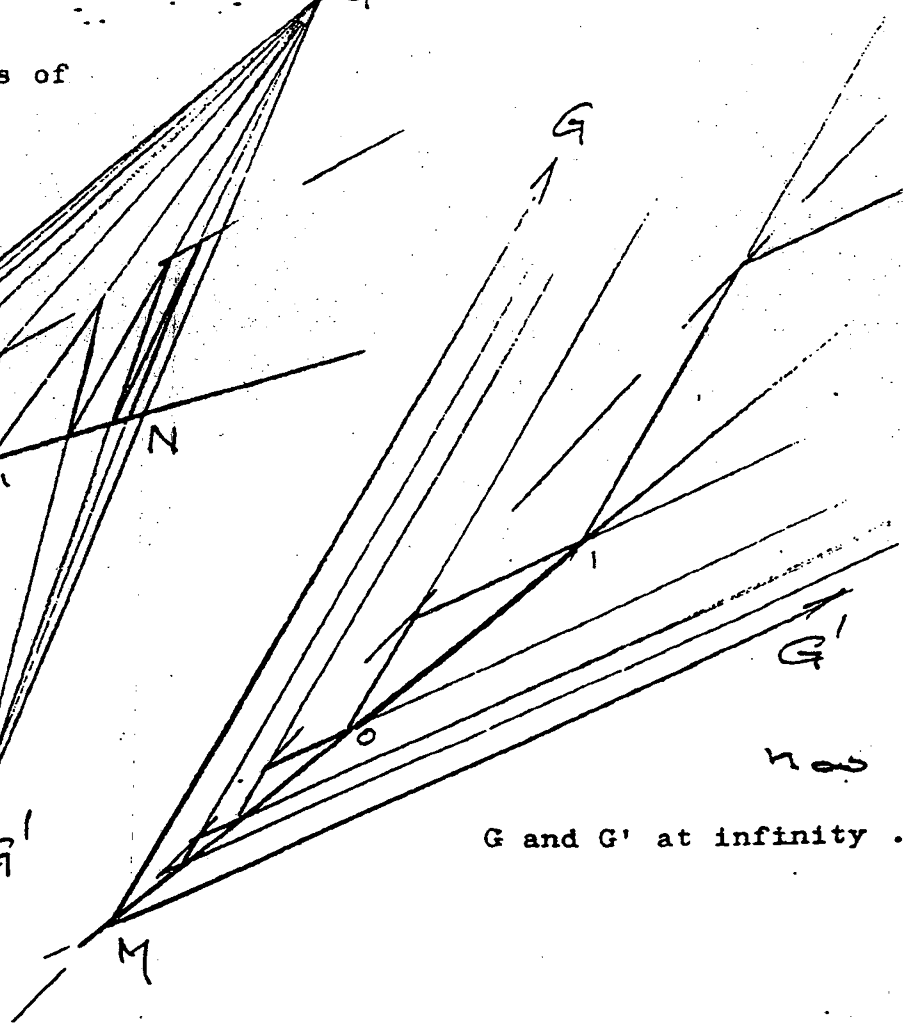
MON $\barwedge$ M1N $\barwedge$ M2N $\barwedge$ M3N ....



Same scale with n and N at infinity.

Further examples of the same scale.



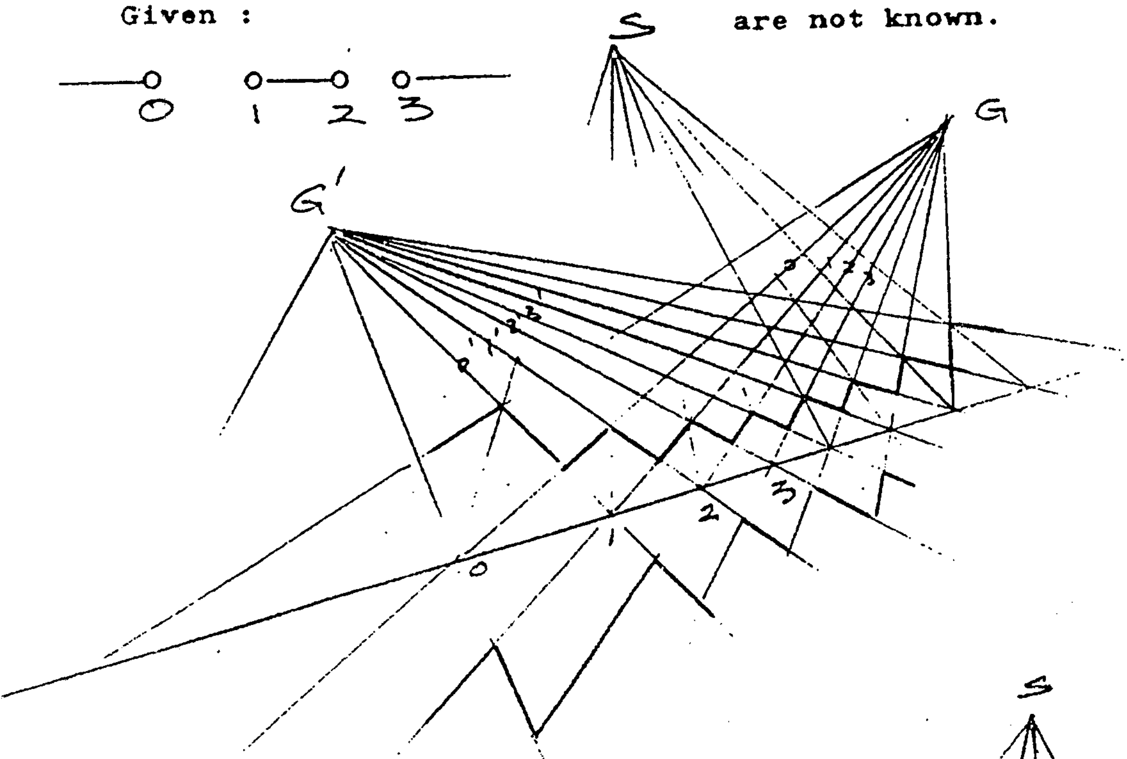


G and G' at infinity.

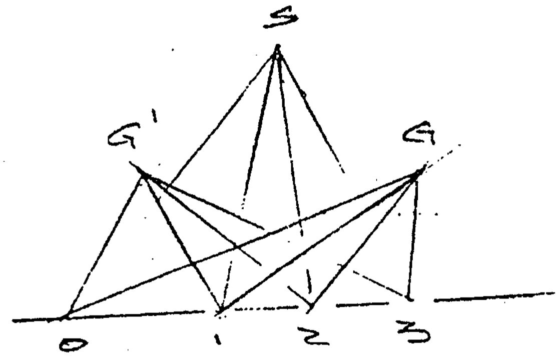
PERIODIC SCALE

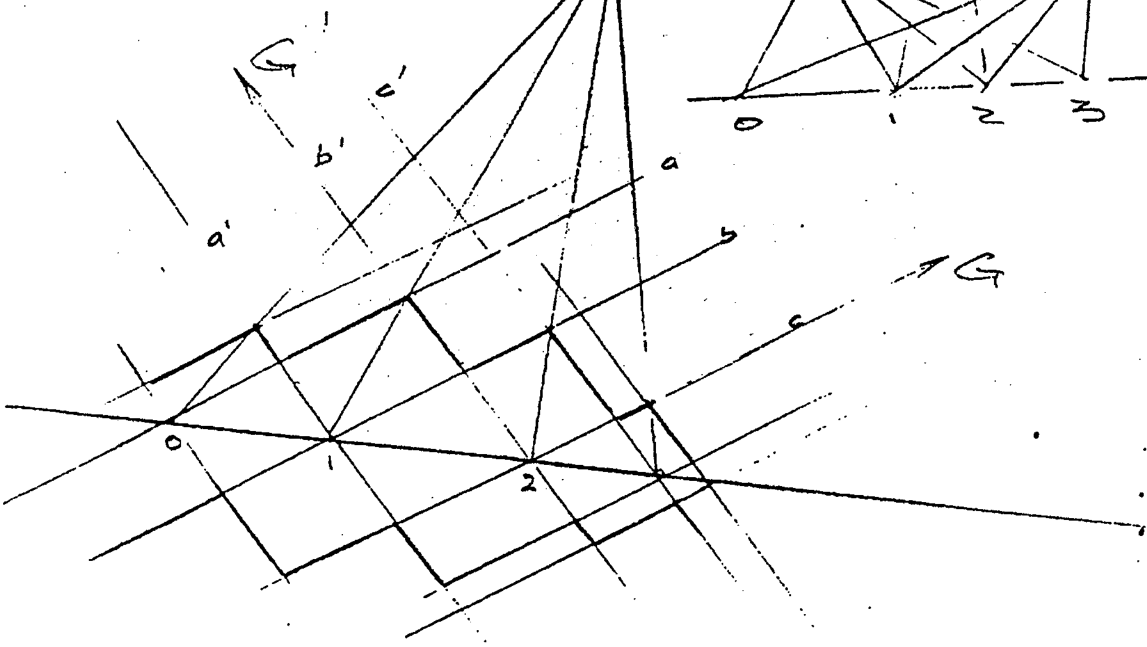
The FIXED POINTS are not known.

Given :



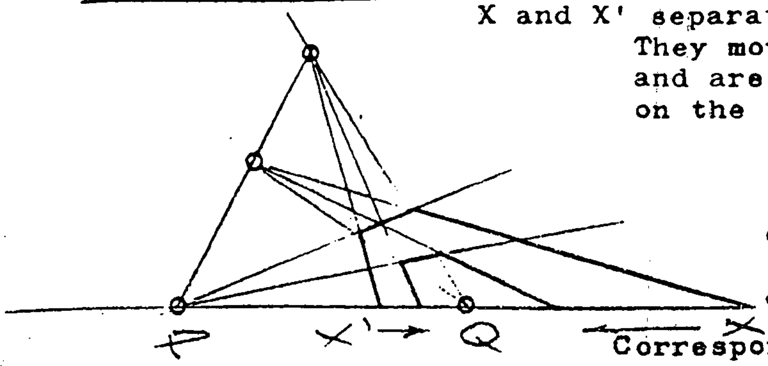
Construction :
o12 $\barwedge$ 123 $\barwedge$ 234 $\barwedge$ 345 ...





G and G' are at infinity.

I N V O L U T I O N

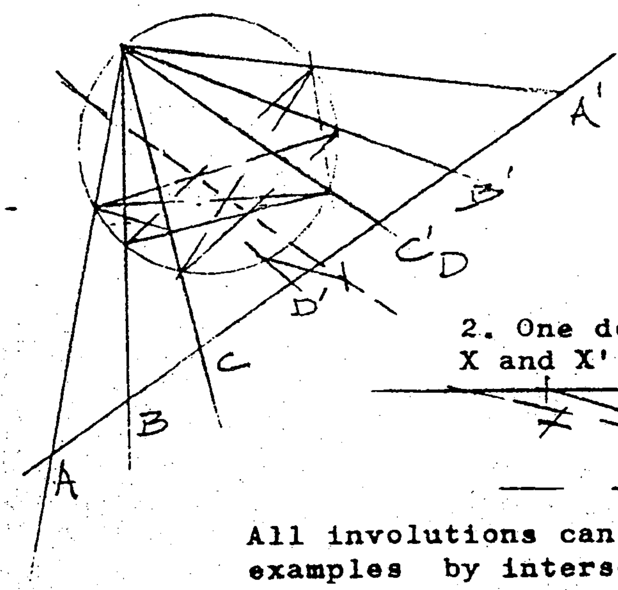


X and X' separate P and Q harmoniously. They move in opposite directions and are two projective ranges on the same line.

P Q are double points.
X as point of range 1 corresponds to X'on range 2.
X as point of range 2 corresponds to X'on range 1.
Corresponding elements are interchangeable.

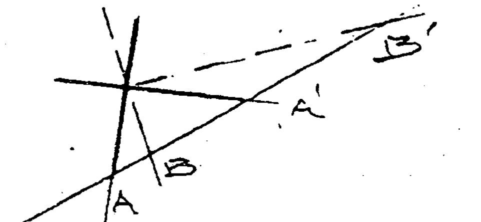
A projectivity between two superimposed ranges or pencils (on the same base) is called an INVOLUTION if the corresponding elements are interchangeable.
(The name involution was given by Desargues.)

A projectivity where the elements are not interchangeable:

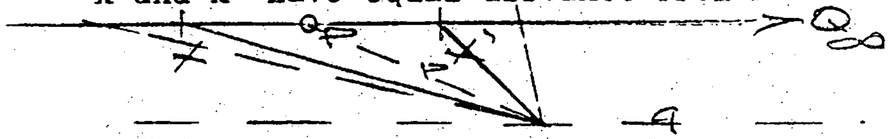


Two examples of Involution:

1. corresponding rays are perpendicular to each other (without double points).



2. One double point is at infinity. X and X' have equal distance from P .



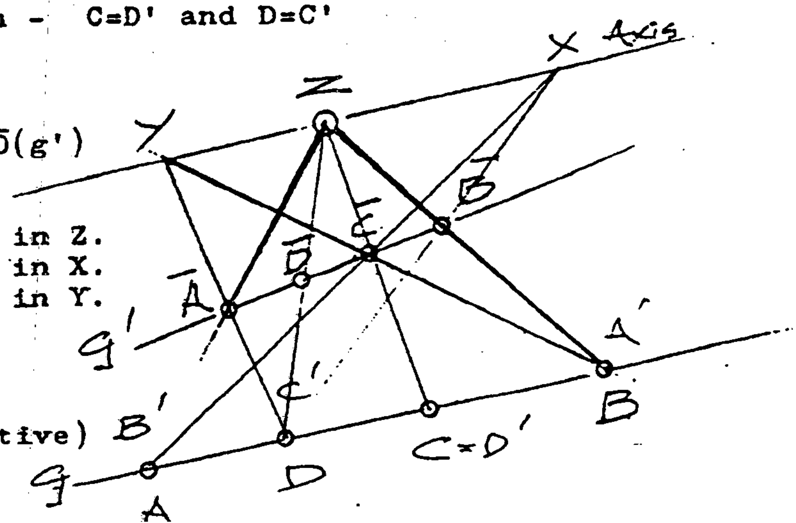
All involutions can be obtained from these two examples by intersection and connection.

If in a projectivity two corresponding elements are interchangeable, all elements are interchangeable.

If A=B' and B=A' - then - C=D' and D=C'

PROOF :
ABC ⊼ A'B'C'
A=B' , B=A'
(g)ABCD⊼(centreZ) $\bar{A}\bar{B}\bar{C}\bar{D}$(g')
(g')$\bar{A}\bar{B}\bar{C}\bar{D}$⊼A'B'C'D'(g)
Axis of perspective:
Lines $\bar{A}$B'+A'$\bar{B}$ intersect in Z.
Lines C'$\bar{B}$+$\bar{C}$B' intersect in X.
Lines $\bar{A}$C'+A'$\bar{C}$ intersect in Y.
Being perspective to Z, C and D interchange, therefore C=D'

(⊼ = symbol for perspective)



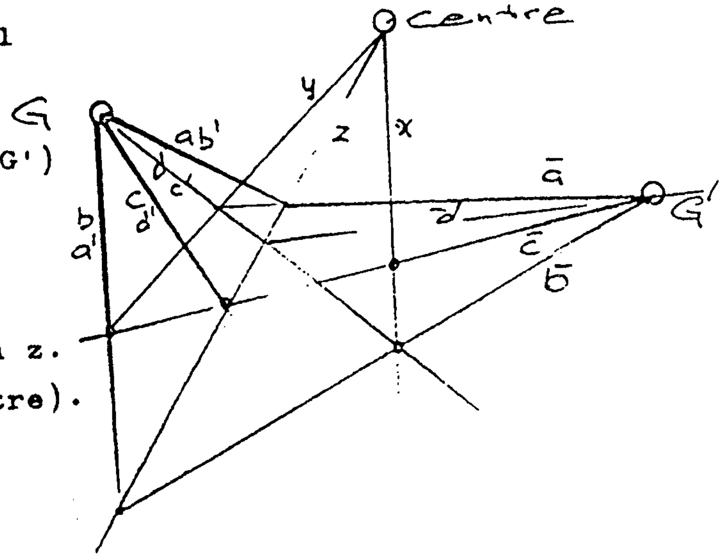
INVOLUTION on a pencil

abc $\barwedge$ a'b'c'
a=b' b=a'
(G)abcd $\barwedge$ (axis z)$\bar{a}\bar{b}\bar{c}\bar{d}$(G')
(G')$\bar{a}\bar{b}\bar{c}\bar{d}$ $\barwedge$ a'b'c'd' (G)
z connects $\bar{a}$b' + a'$\bar{b}$.
x connects c'$\bar{b}$ + $\bar{c}$b'.
y connects $\bar{a}$c' + a'$\bar{c}$.

dc' + d'c intersect on z.

x y z concurrent (centre).

(If two elements are corresponding then all are.)

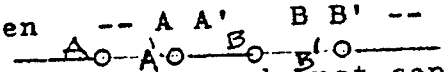


Two pairs of corresponding elements determine one and only one involution.

The HYPERBOLIC INVOLUTION has two double elements.

The ELLIPTIC INVOLUTION has no double elements.

Example 1: Two pairs are given -- A A' B B' --

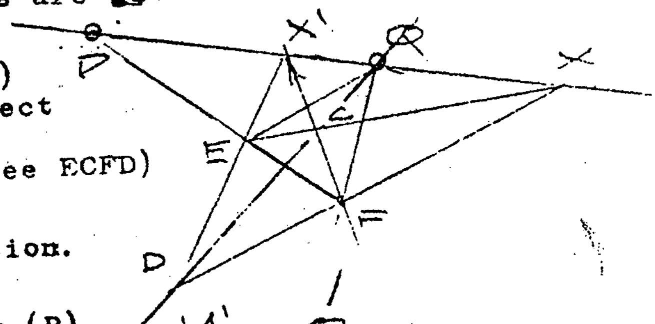


AA'B $\barwedge$ A'AB' in opposite directions. Pairs do not separate each other. It is a hyperbolic involution.

Example 2: Two double elements are given -- P Q --

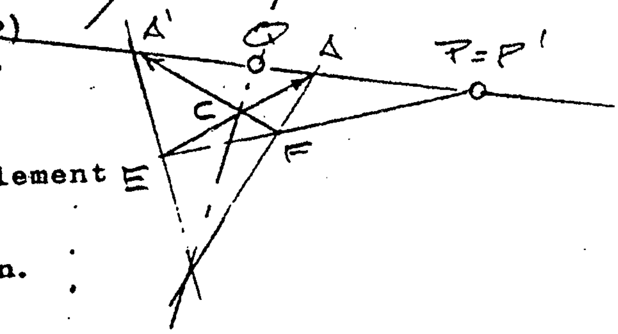
PQXX' $\barwedge$ PQX'X
E(PQXX') $\barwedge$ F(PQX'X)
Corr. lines intersect on the axis QCD.
PQ-XX' are harm. (see ECFD)

It is a hyperbolic involution.



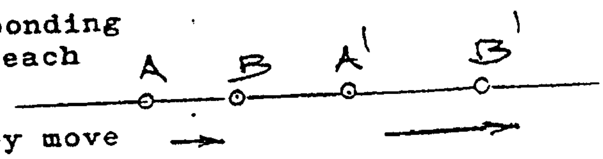
Example 3: One double element (P) and one pair of corr. elements are given.

A'AP $\barwedge$ AA'P.
Q is second double element
E(PAA') $\barwedge$ F(P'A'A)
Axis CDQ . This is a hyperbolic involution.

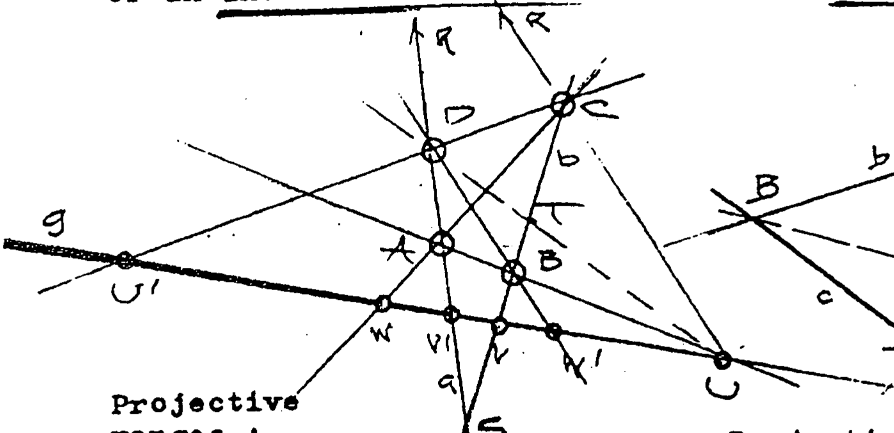


Example 4: Two pairs of corresponding elements separating each other are given.

ABA'B' $\barwedge$ A'B'AB. They move in the same direction- this is an elliptic involution .

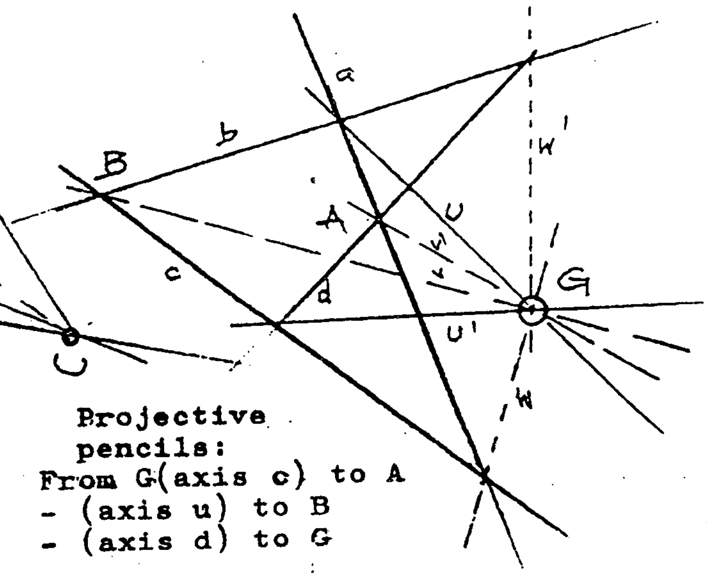


Complete QUADRANGLE	Complete QUADRILATERAL
-opposite sides intersect a line in corresponding points of an involution.	- opposite points connect with a point in corresponding rays of an involution.



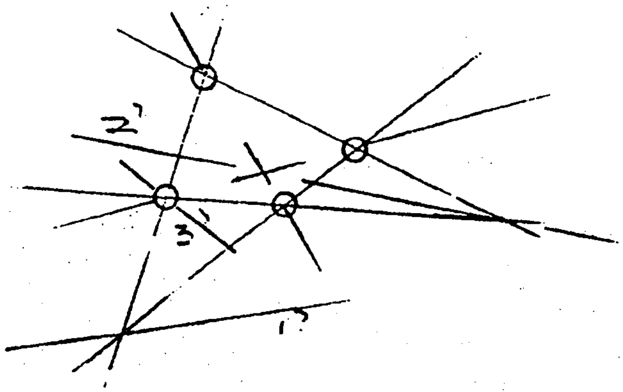
Projective ranges :

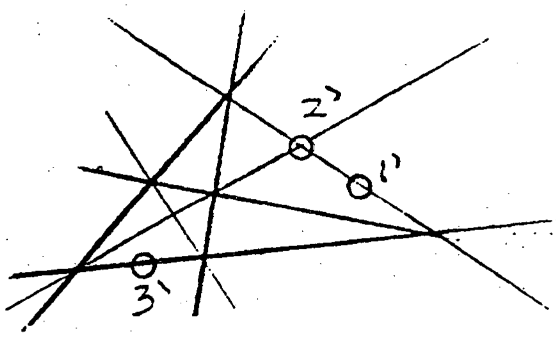
From g: UU'VV'W (centre C)
to a: RSDV'A (centre U)
to b: CTSVB (centre D)
to g: U'UV'VW'



Projective pencils:

From G(axis c) to A
- (axis u) to B
- (axis d) to G

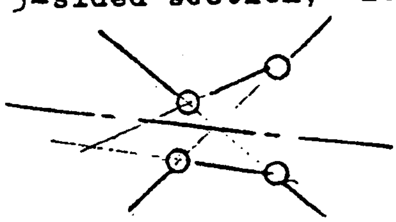




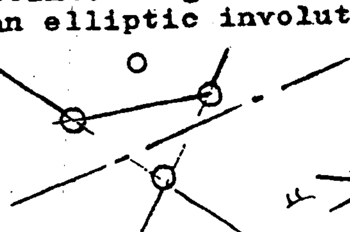
	- On a line -	- Through a point -	the involution
1)	through 1 extra pt.	on an extra side	has 2 double elements
2)	through 2 extra pts.	on 2 extra sides	is hyperbolic.
3)	through 1 corner	on 1 side	is parabolic.

If the line is in the 4-pointed region or the point in the 4-sided section , it is a hyperbolic involution.

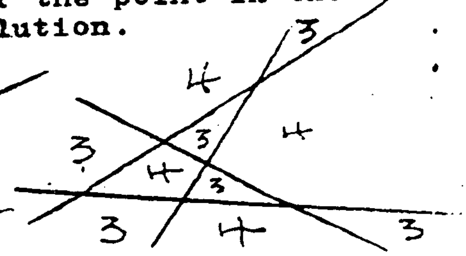
If the line is in the 3-pointed region or the point in the 3-sided section, it is an elliptic involution.



4-pointed region (hyperbolic)

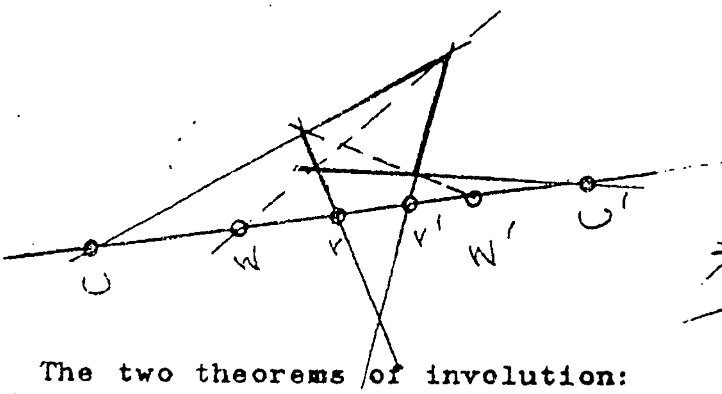


3-pointed region (elliptic)

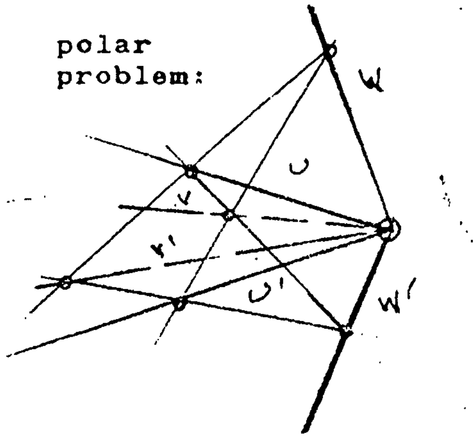


Sections-numbered according to their number of sides.

Problem: An involution is given by two pairs of corresp. points UU' ,VV'. Construct,with a quadrangle to an arbitrary point W,the corresponding W'.



polar problem:



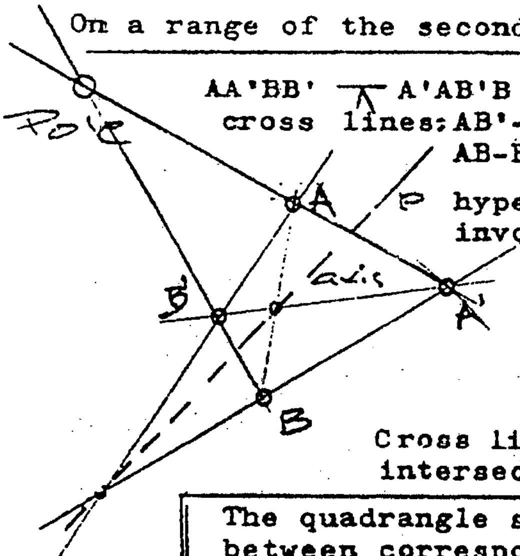
The two theorems of involution:

1. If one pair of elements is interchangeable, all are interchangeable.
2. Two pairs of elements determine an involution.

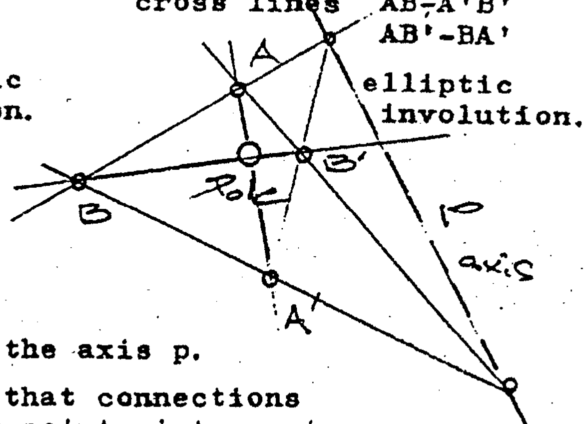
Both theorems are also valid for an involution of the second degree (on a conic) because an involution on a first degree base can be transferred to a conic by perspective.

On a range of the second order :

AA'BB' ⊼ A'AB'B
cross lines: AB'–BA'
AB–B'A'
hyperbolic involution.



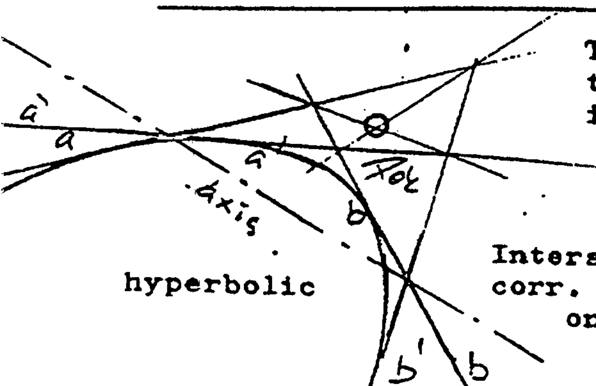
AB'A'B ⊼ A'BAB'
cross lines AB–A'B'
AB'–BA'
elliptic involution.



Cross lines intersect on the axis p.

The quadrangle shows that connections between corresponding points intersect in the pole. (P is the centre of the involution.)

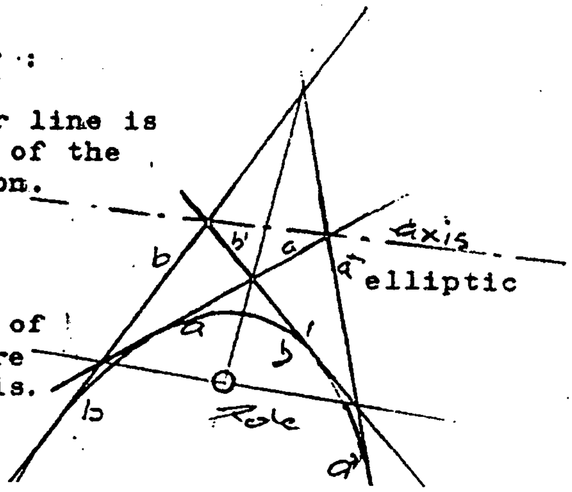
On a pencil of the second order :



The polar line is the axis of the involution.

hyperbolic

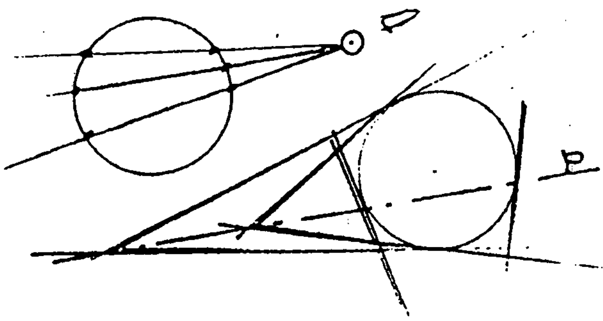
Intersections of corr. lines are on the axis.

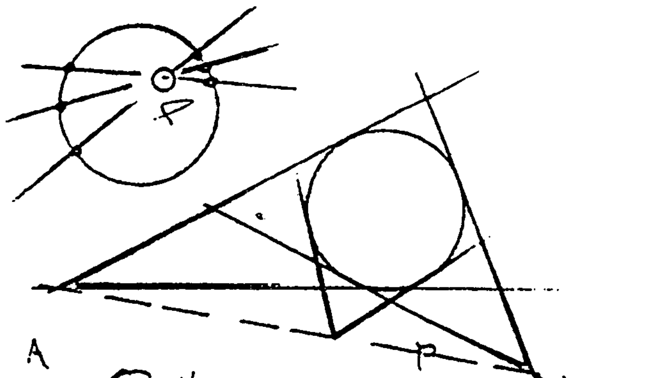


elliptic

Hyperbolic Involution | Elliptic Involution

Corresponding points are on lines through point P.
Corresponding lines (tangents) intersect on line p.

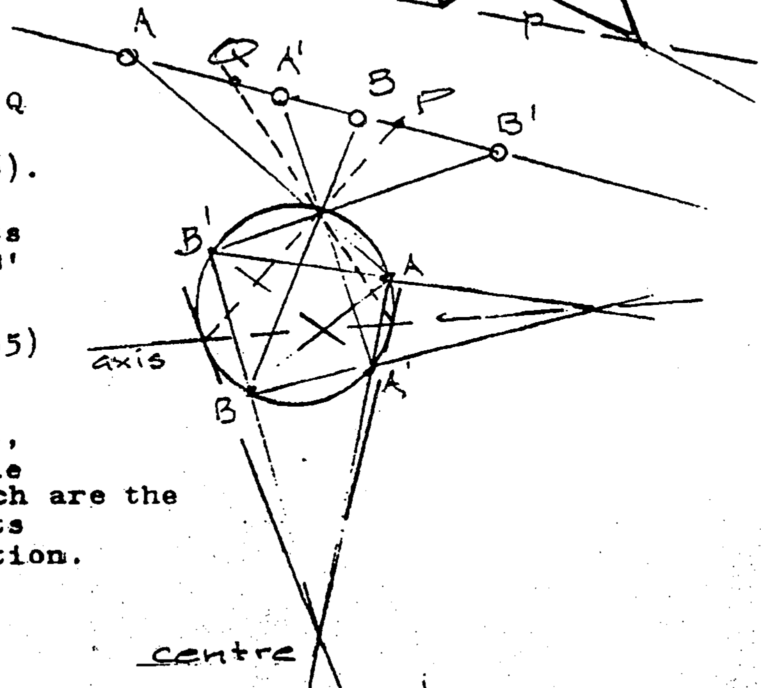




Given:
Two pairs of points
not separating each other.
Construction for points P,Q
which separate both pairs
harmoniously (See page 26).

P Q are the double elements
of the involution A A' B B'

Using J.Steiner's double
point construction (page 35)
transfer to circle:
AA'BB' $\barwedge$ A'AB'B .
Cross points give the axis,
which intersects the circle
in the double points, which are the
touching points of tangents
from the centre of involution.



HARMONIOUS REPRESENTAION
of an ELLIPTIC INVOLUTION

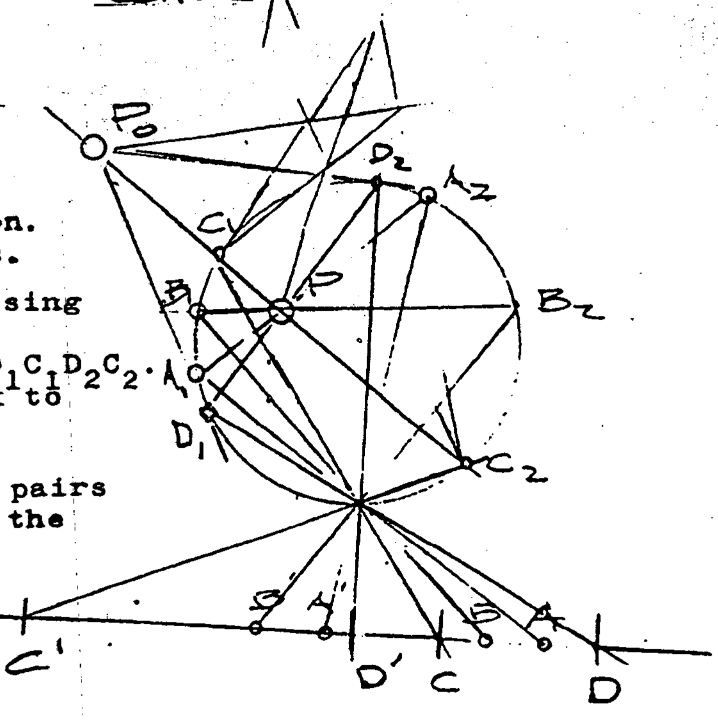
AA'BB' separating pairs-
transfer to circle.

P - centre of the involution.
$P_o P$ $C_1 C_2$- harm. four points.

Harmonious pencil in D_1 passing
through $P_o C_1 P$ C_2
intersects with circle in $D_1 C_1 D_2 C_2$.
Transfer latter points back to
the line g.

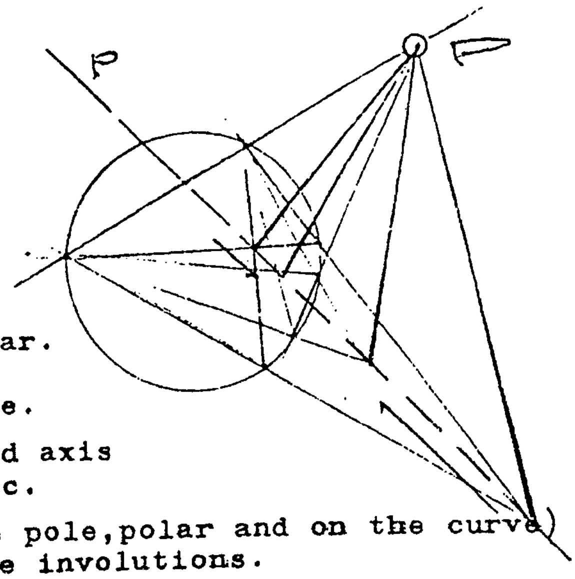
D C D' C' are therefore two pairs
of corresponding points of the
elliptic involution
in harmonious sequence.

Point C can be chosen
anywhere on the line.



H4

IMAGINARY ELEMENTS



Conjugate elements are corresponding elements of an involution.

A conic curve creates an involution in every point and every line in the plane of this conic.
P,p outside: hyperbolic
inside : elliptic

To every point belongs a definite line -- the polar.
to every line belongs a definite point -- the pole.

Pole and polar are centre and axis of the involution on the conic.

The three involutions (in the pole,polar and on the curve) are determined by one of these involutions.

The double elements are points or tangents of the curve (conic curve of second degree).

The hyperbolic involutions each produce two real points,and two real tangents by their double elements.

The ELLIPTIC INVOLUTION is considered as the representation of two I M A G I N A R Y P O I N T S
or two I M A G I N A R Y L I N E S

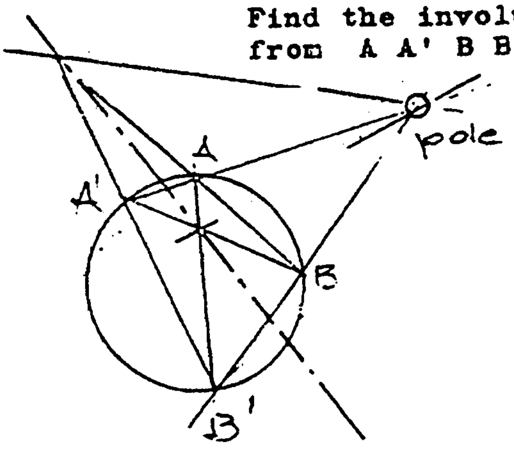
An imaginary element is represented by a directed elliptic involution.
(von Staudt 1847. 1798-1867)

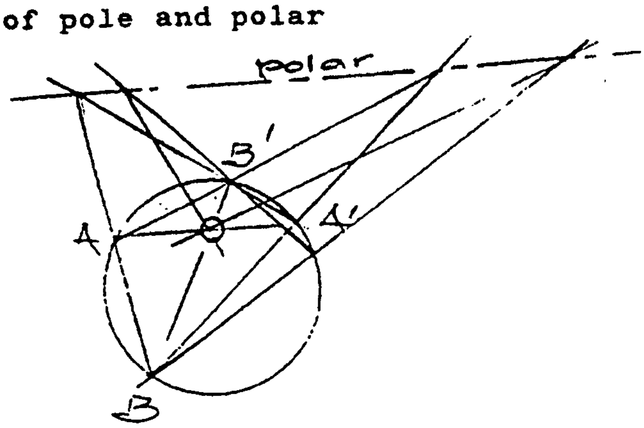
Every line in the plane of a conic has two points in common with this conic and every point in this plane has two tangents in common with this conic; two real or two imaginary ones.

We can consider the elliptic involution to be a movement - in one direction as being one imaginary element, and in the other direction as a second imaginary element. In the same way we declare the imaginary plane to be a movement in a pencil of planes.

The concepts point, line and plane are thereby extended.

Find the involutions of pole and polar from A A' B B'

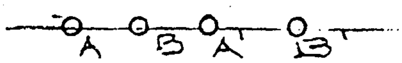




The phenomena of intersecting and connecting between the elements of point, line and plane are valid also for the extended concepts of point, line and plane, i.e. the imaginary elements. (Abbr. im.=imaginary)

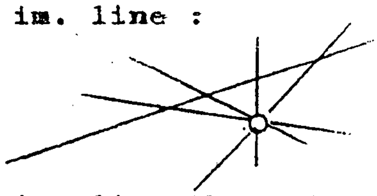
For the geometry of the plane

An im. point
on a real line :

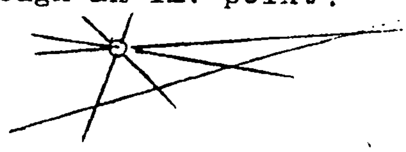


An im. line
through a real point :

An im. point
on an im. line :



An im. line
through an im. point :



On an im. line there is only one real point.
Through an im. point is only one real line.
Through 2 points there is one and only one line.
Two lines have one and only one point in commom.

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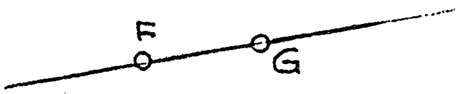
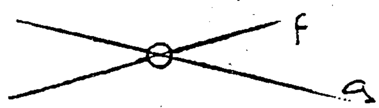
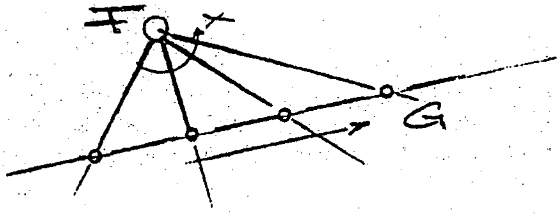
1. F, G both real points



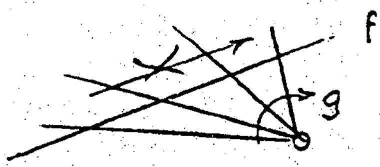
1. f, g both real lines



2. F real, G im.
Connection x im.



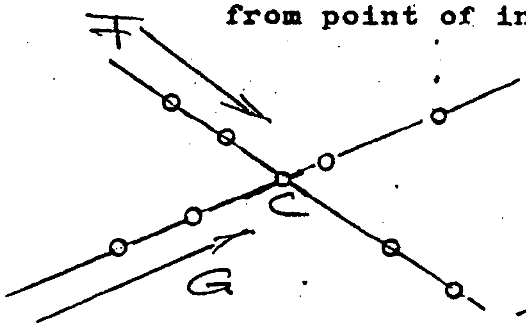
2. f real, g im.
Intersection X im.

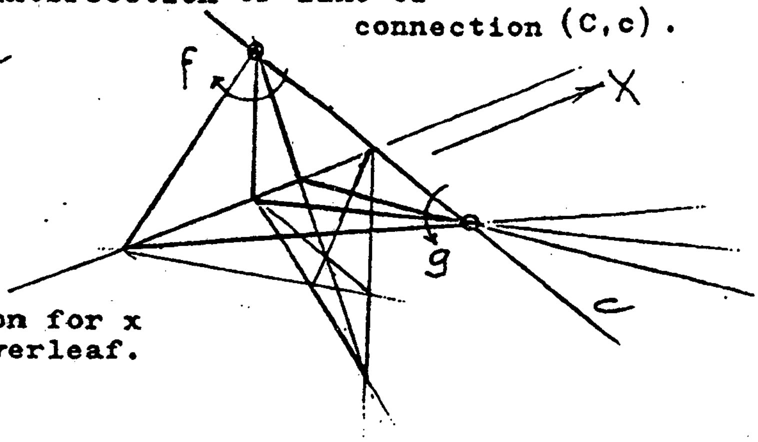


3. F im., G im.
Connection x im.

3. f im., g im.
Intersection X im.

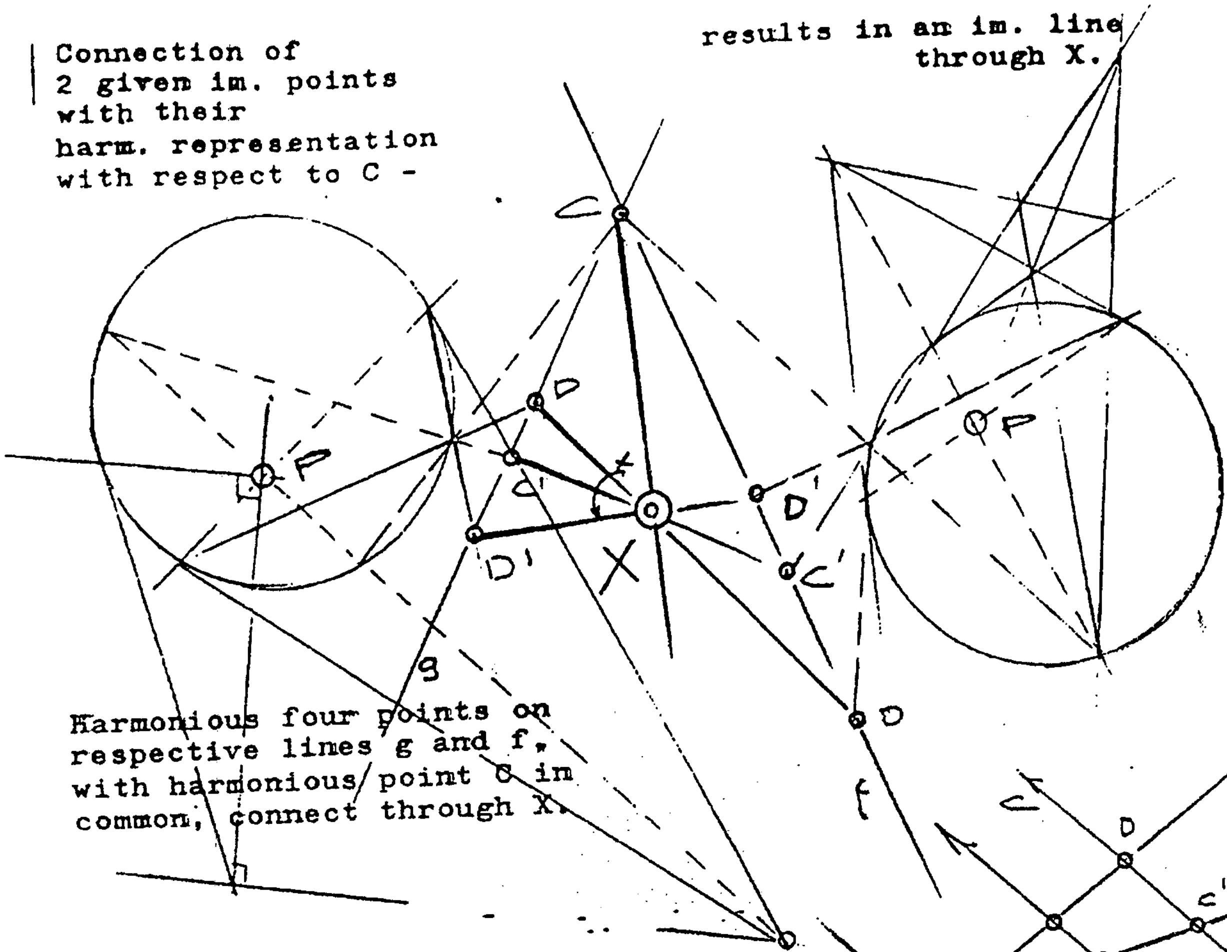
Construction with harmonious representation
from point of intersection or line of
connection (C,c) .



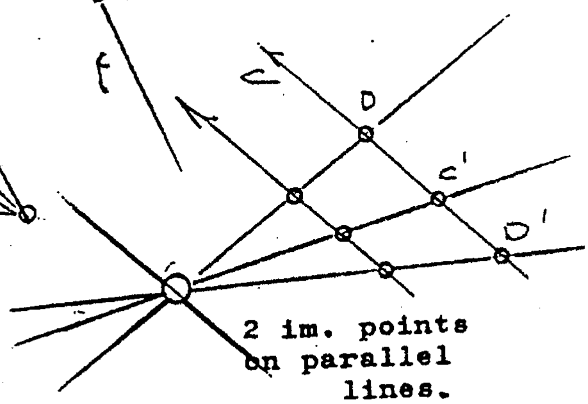


Construction for x
and X is overleaf.

Connection of 2 given im. points with their harm. representation with respect to C - results in an im. line through X.

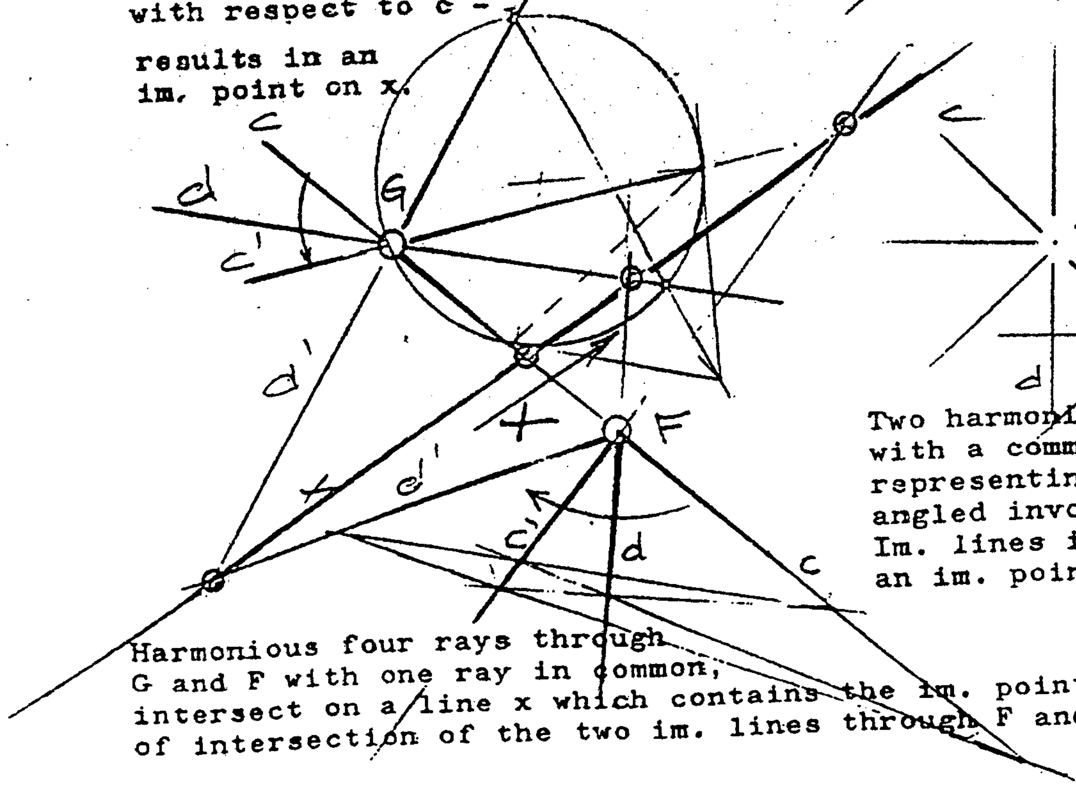


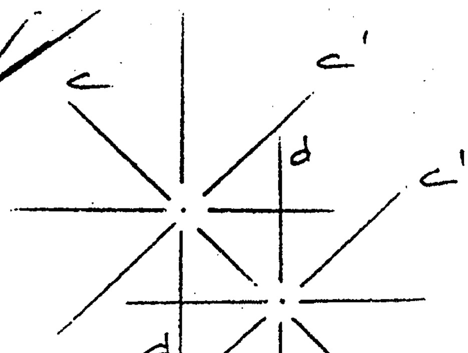
Harmonious four points on respective lines g and f, with harmonious point C in common, connect through X.



2 im. points on parallel lines.

Intersection of 2 given im. lines with their harm. representation with respect to c - results in an im. point on x.



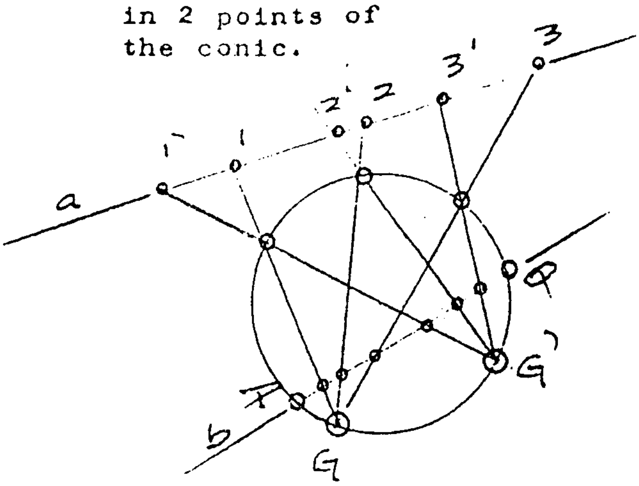


Two harmonious four rays with a common ray, representing two right-angled involutions. Im. lines intersect on an im. point on the line at infinity.

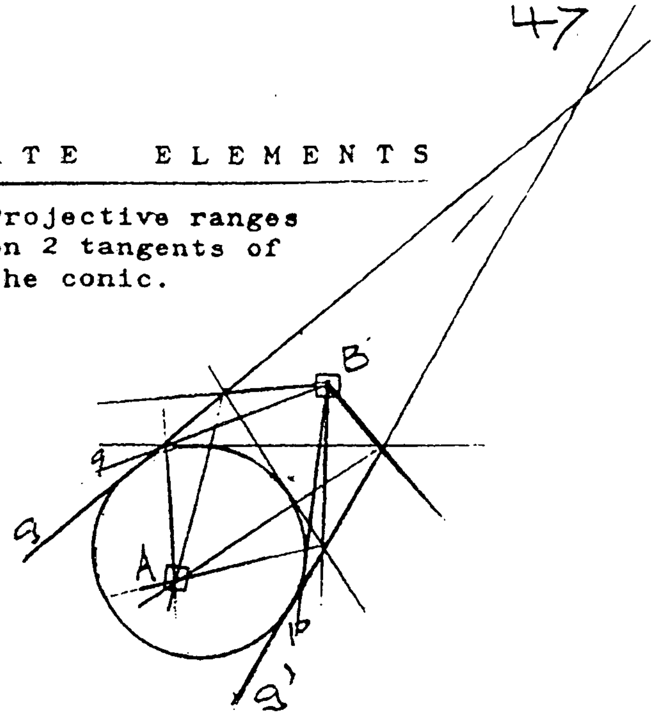
Harmonious four rays through G and F with one ray in common, intersect on a line x which contains the im. point X of intersection of the two im. lines through F and G.

Construction of CONJUGATE ELEMENTS

Projective pencils in 2 points of the conic.

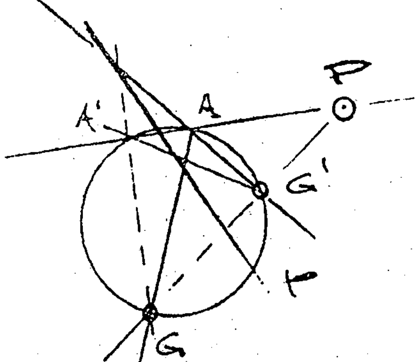


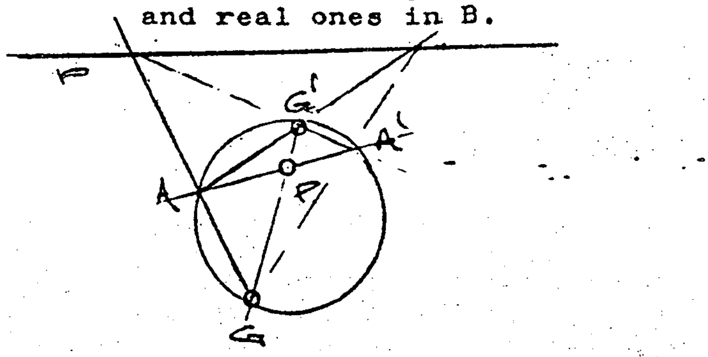
Projective ranges on 2 tangents of the conic.



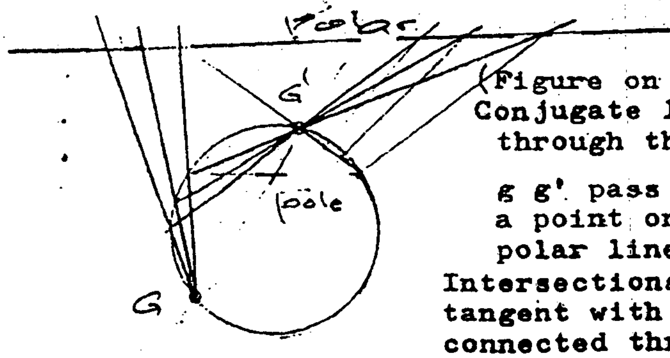
The 2 pencils produce 2 ranges on a or b with im. double points on a and real ones on b.

The 2 ranges produce 2 pencils in A or B with im. double tangents in A and real ones in B.





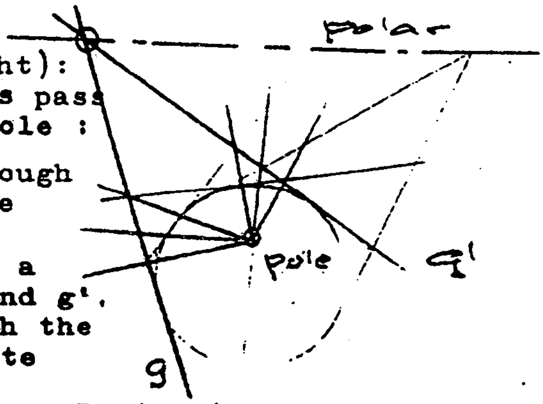
Pole P produces an involution on the conic curve. P is the centre of involution : corresponding points lie on the line through P (AA'). The corresponding involution lies on the polar p(constructed with a quadrangle) GG' being on a line through the pole. The connections of any point on the conic with G and G' intersect the polar in 2 conjugate points. (Figures above and below left).



(Figure on right): Conjugate lines pass through the pole :

g g' pass through a point on the polar line.

Intersections of a tangent with g and g', connected through the pole are conjugate pairs of lines.



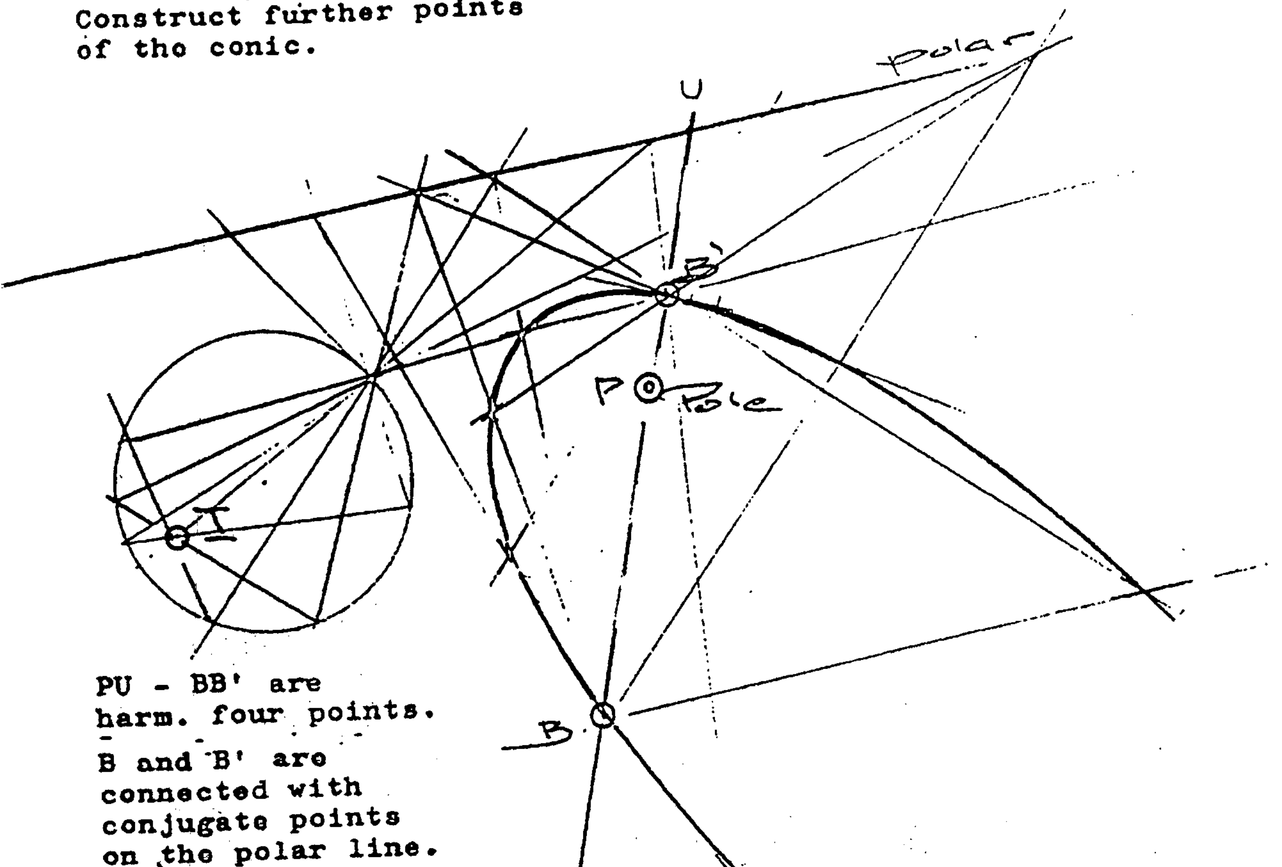
Projective pencils occur in G and G'.

Projective ranges occur on g and g'.

Problem

given: Pole and
polar line with
involution and point B
a point of the conic.
Construct further points
of the conic.

The involution on the
polar is produced by an
auxiliary circle
and a centre I of the
involution .

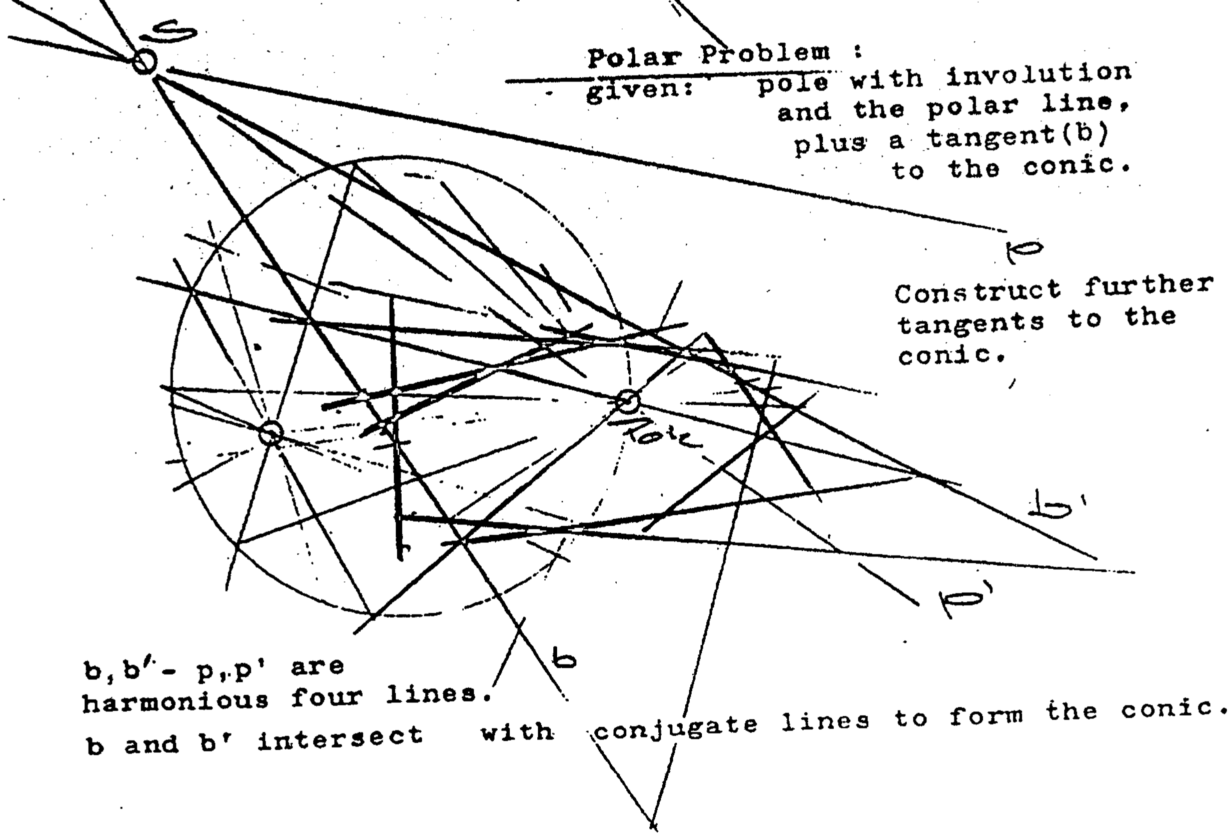


PU - BB' are
harm. four points.
B and B' are
connected with
conjugate points
on the polar line.

Polar Problem :

given: pole with involution
and the polar line,
plus a tangent (b)
to the conic.

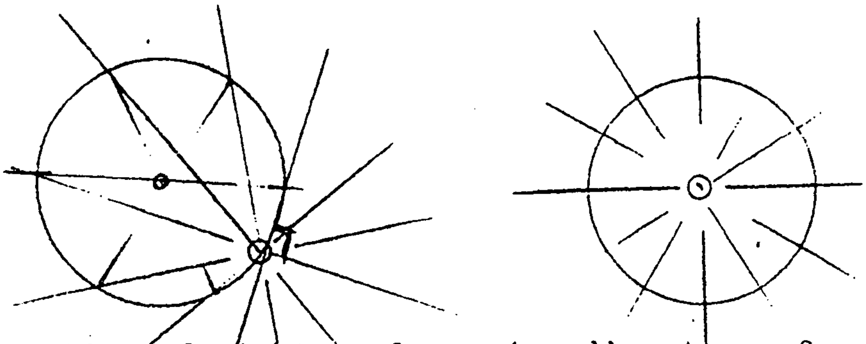
Construct further
tangents to the
conic.



b, b' - p, p' are
harmonious four lines.
b and b' intersect with conjugate lines to form the conic.

The ABSOLUTE INVOLUTION

on the line at infinity is produced by
a right-angled ray involution.

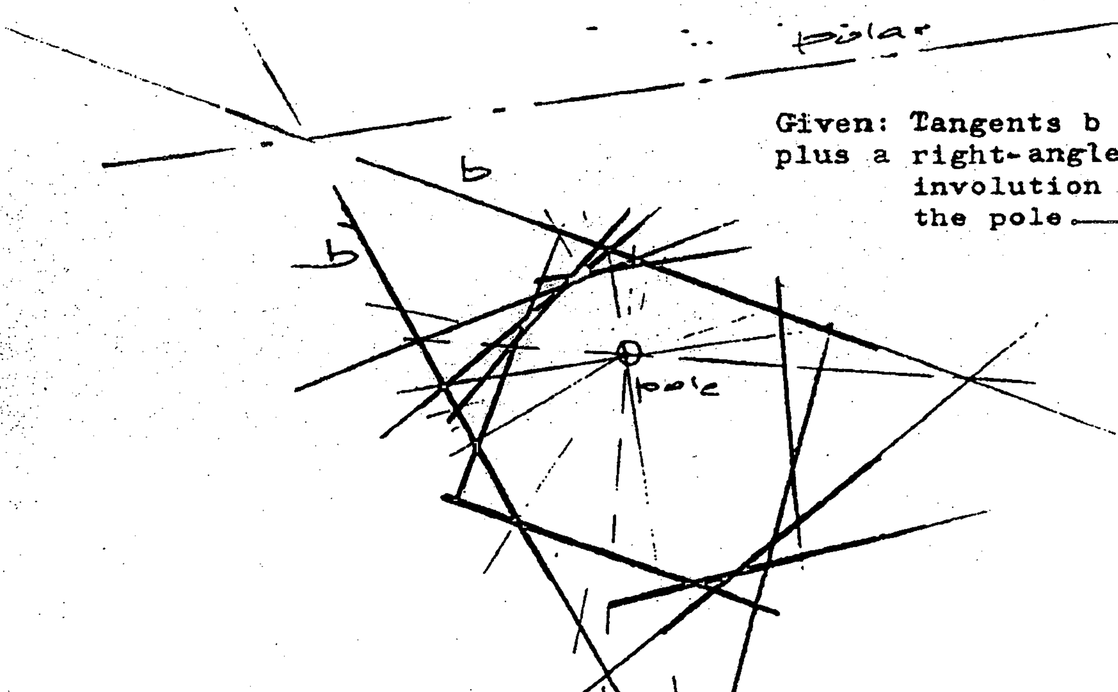


Construction of right angles using diameters of a circle.

In the centre of a circle is a right-angled ray involution.
Conjugate lines are then perpendicular to each other.

Every circle produces
an absolute involution on the line at infinity.

The absolute involution on the line at infinity is an elliptic involution and represents the imaginary points of intersection, in which every circle intersects the line at infinity. They are simply called
the ABSOLUTE POINTS.

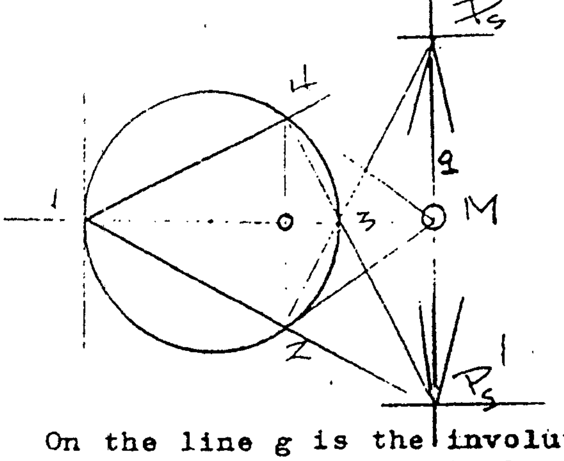


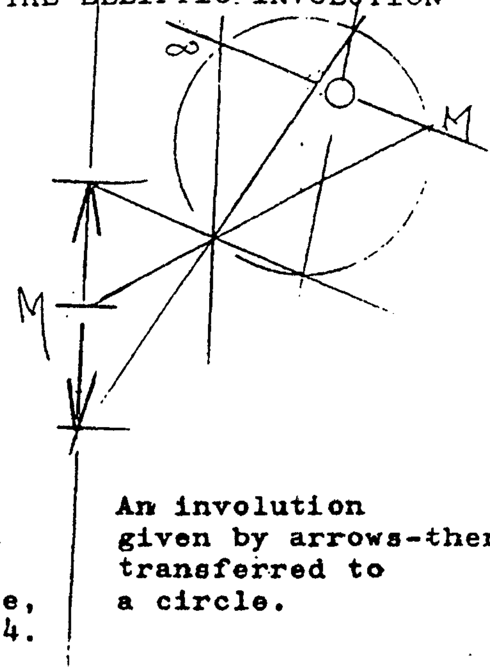
Given: Tangents b b',
plus a right-angled involution in the pole.

The imaginary lines of a right-angled ray involution
are called MINIMAL LINES ("Minimale oder
or ISOTROPE LINES Isotrope Geraden")

They are the two tangents to a circle intersecting in the centre of the circle.

ARROW REPRESENTATION OF THE ELLIPTIC INVOLUTION

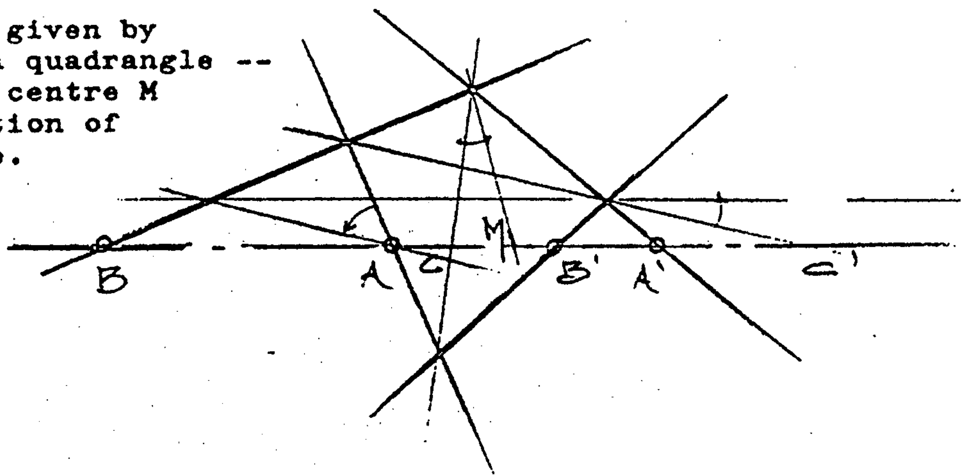


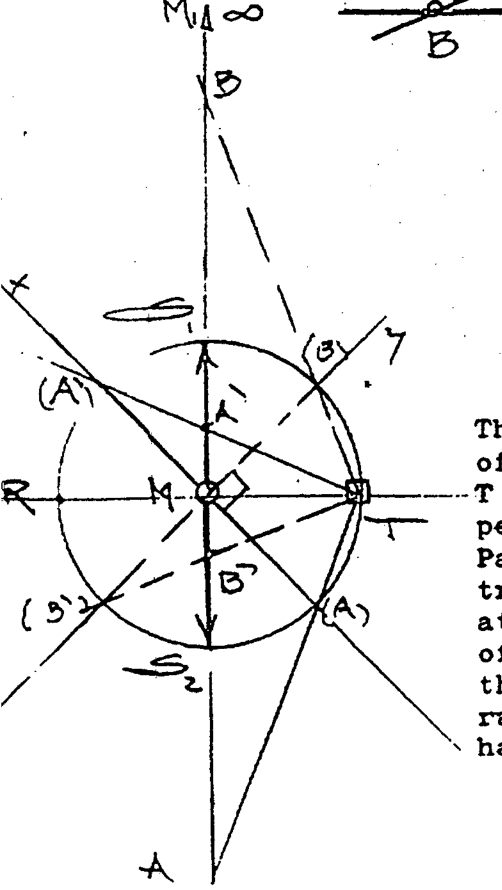


On the line g is the involution with respect to the circle. Conjugate pairs are: the centre M, the point at infinity, and there is a symmetrical pair $P_s - P_s'$ with equal distances from the centre, constructed with the quadrangle 1234. The two arrows represent the two imaginary points.

An involution given by arrows—then transferred to a circle.

An involution given by the sides of a quadrangle -- construct the centre M by transformation of the quadrangle.

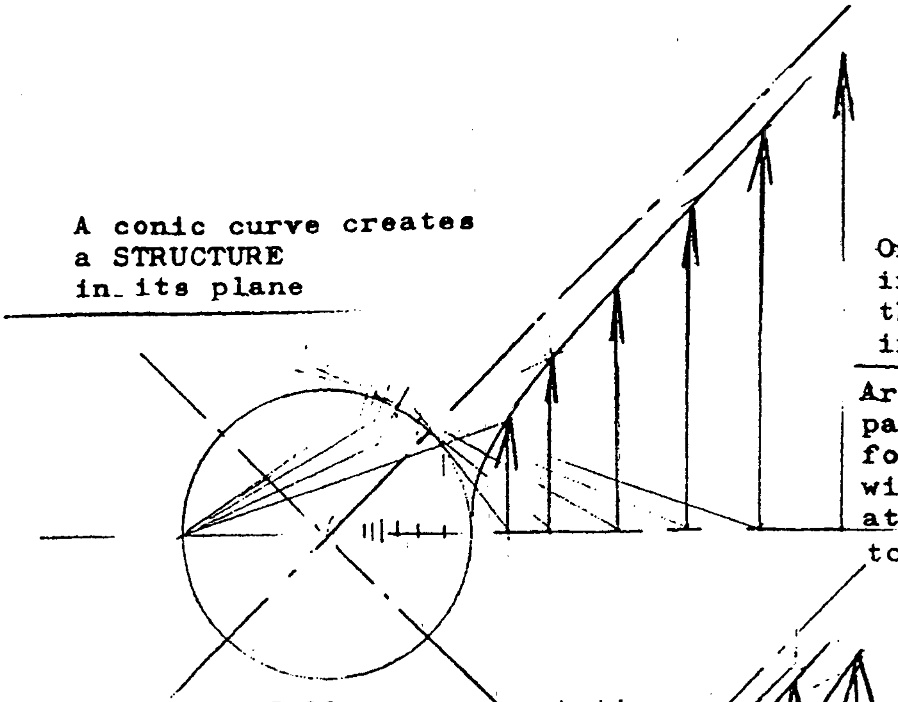




An elliptic involution is here given with arrows. Shown is the construction of any conjugate pair, and harmonious representation with respect to a point A.

The circle passes through the points of the arrows, with M as the centre. T is at the intersection of circle with perpendicular to arrows through M. Pairs S_1S_2 and M(M_1 at infinity) are transferred through T to the circle at S_1S_2 and R T. M is thus the centre of involution. To any point A construct the corresponding point A'. Right-angled ray y(to x) produces the pair BB' harmonious to AA'.

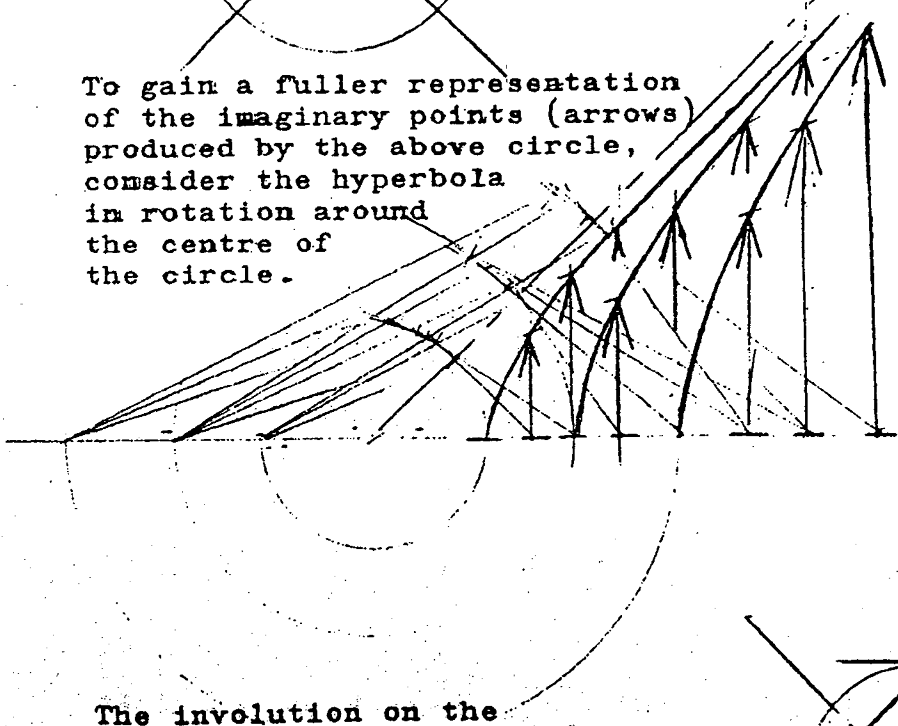
A conic curve creates
a STRUCTURE
in its plane



On every line and
in every point in
the plane is an
involution.

Arrows on
parallel lines here
form a hyperbola
with asymptotes
at right angles
to each other.

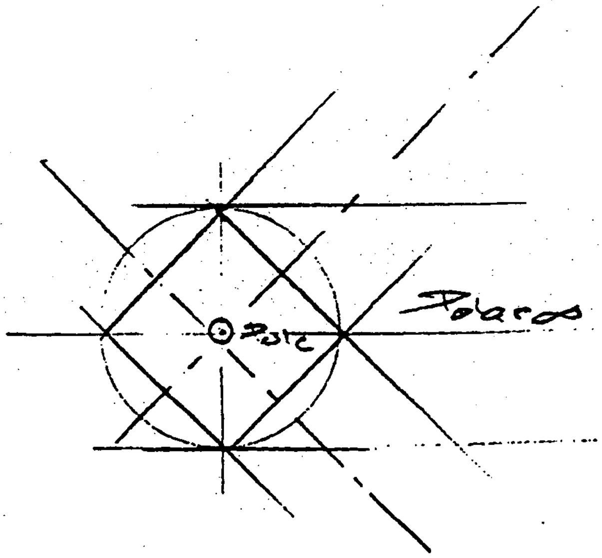
To gain a fuller representation
of the imaginary points (arrows)
produced by the above circle,
consider the hyperbola
in rotation around
the centre of
the circle.



Concentric circles
produce a family
of hyperbolas.

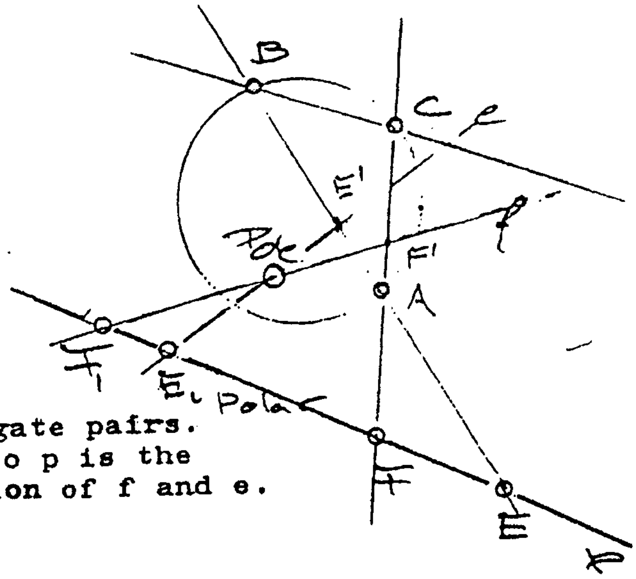
The involution on the
line at infinity with
respect to any circle is
right-angled, and is the
absolute involution.

Two lines are at right angles,
if they intersect the line at
infinity in conjugate points
of the absolute involution.



Every circle intersects the line at infinity in the
two absolute points (elliptic involution).
In the centre of the circle are two imaginary tangents
(the absolute line-involution or the "minimal" lines).

Problem
given : 3 real + 2 im. points of a conic, construct further points of the conic.

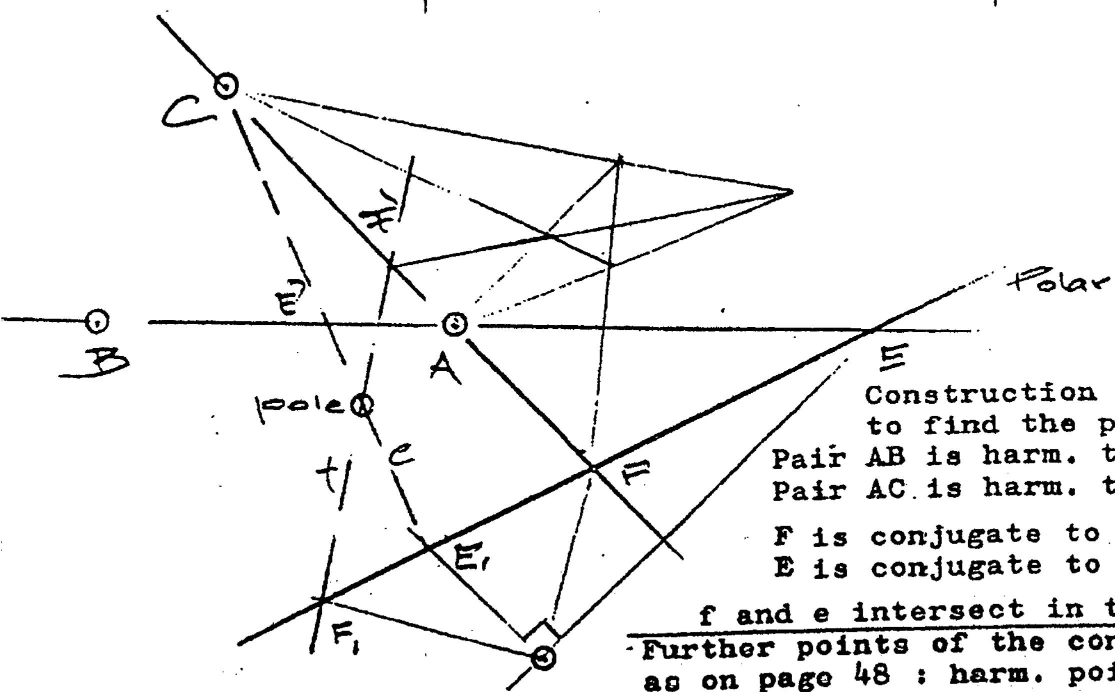


Conjugate points are on each other's polar line.

All points on a polar line are conjugate to the pole.

e is polar to E
f is polar to F
CA are harm. to F'F
BA are harm. to E'E

EE_1 FF_1 are conjugate pairs.
The pole to p is the intersection of f and e.



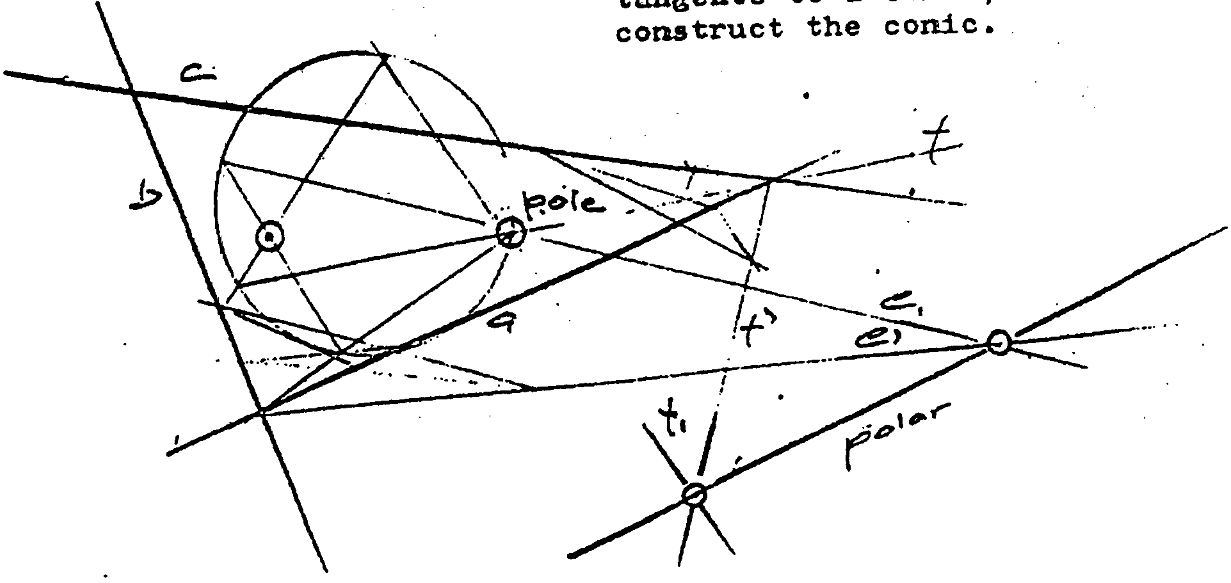
Construction to find the pole:
Pair AB is harm. to EE'
Pair AC is harm. to FF'

F is conjugate to F_1
E is conjugate to E_1

f and e intersect in the pole.

Further points of the conic found as on page 48 : harm. point is on a line through the pole, etc.

Polar problem :
given 3 real + 2 im. tangents to a conic, construct the conic.

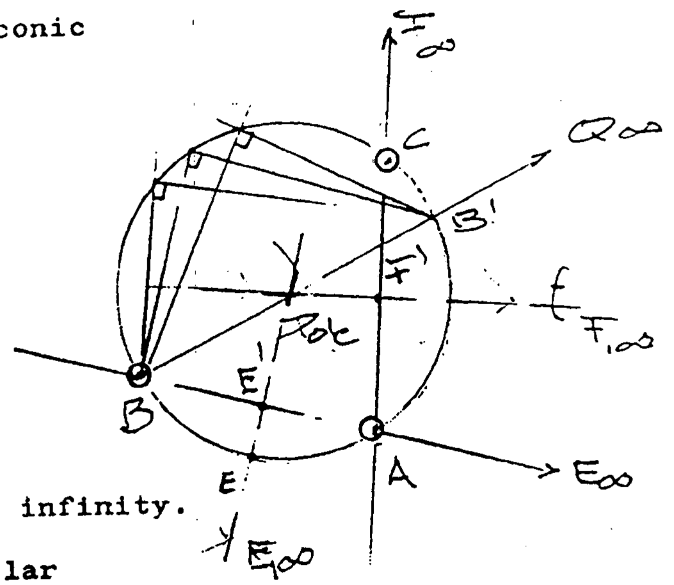


Problem : construct the conic given 3 real points ABC and the 2 absolute points (polar line at infinity).

Pair AC is harm. to F'F
Pair AB is harm. to E'E

Conjugate points are:
F, F_1 E, E_1 at infinity
F' F_1 polar to F
E' E_1 polar to E

The pole is on the two bisecting perpendiculars to A C and A B ---

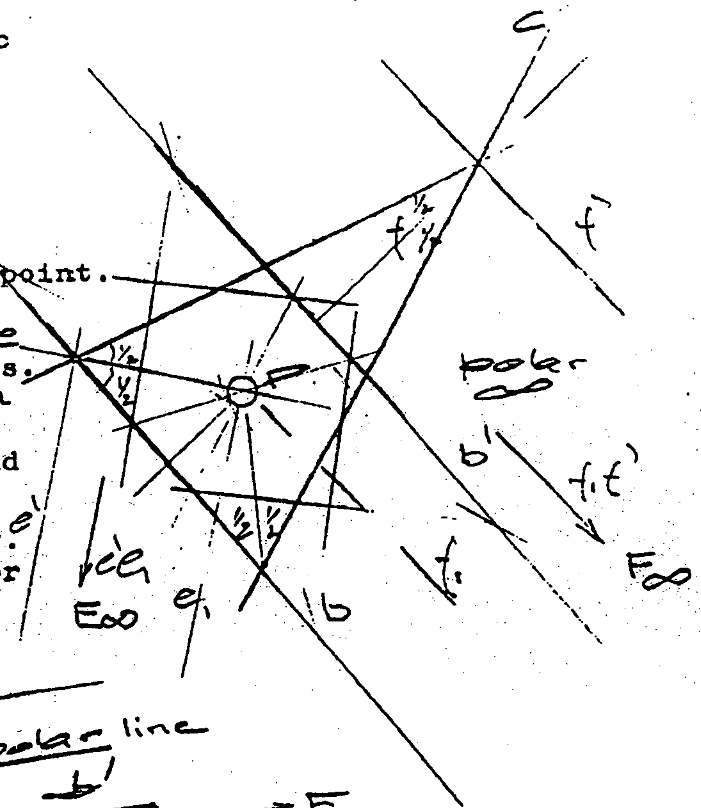


B'B harm. to pole and Q at infinity. From B and B' lines to the conjugate points on the polar (right-angled) intersect in points on a circle (the conic to be found).

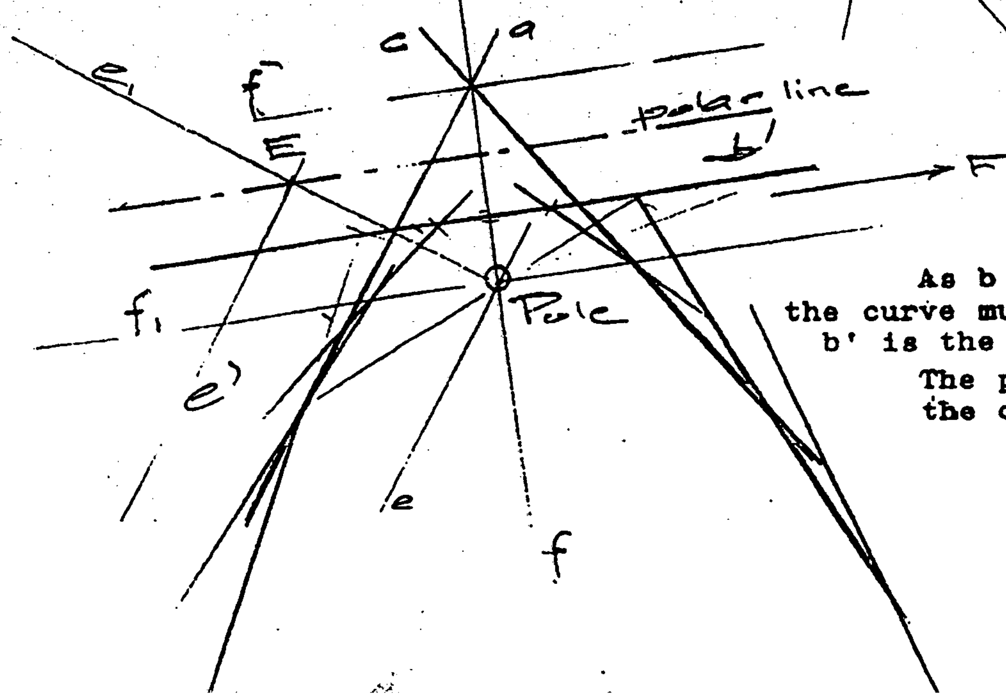
Polar problems:
given 3 tangents abc and pole with absolute involution, find the conic if :

1. P is the angle-bisecting point.
   Pair ee' is harm. to ab
   Pair ff' is harm. to ac
   ee_1 and ff_1 are conj. pairs.

   F E is polar line.
   Further tangents are found between b and b'.



2. b is the line at infinity.
   P is on the angle bisector between a and c .



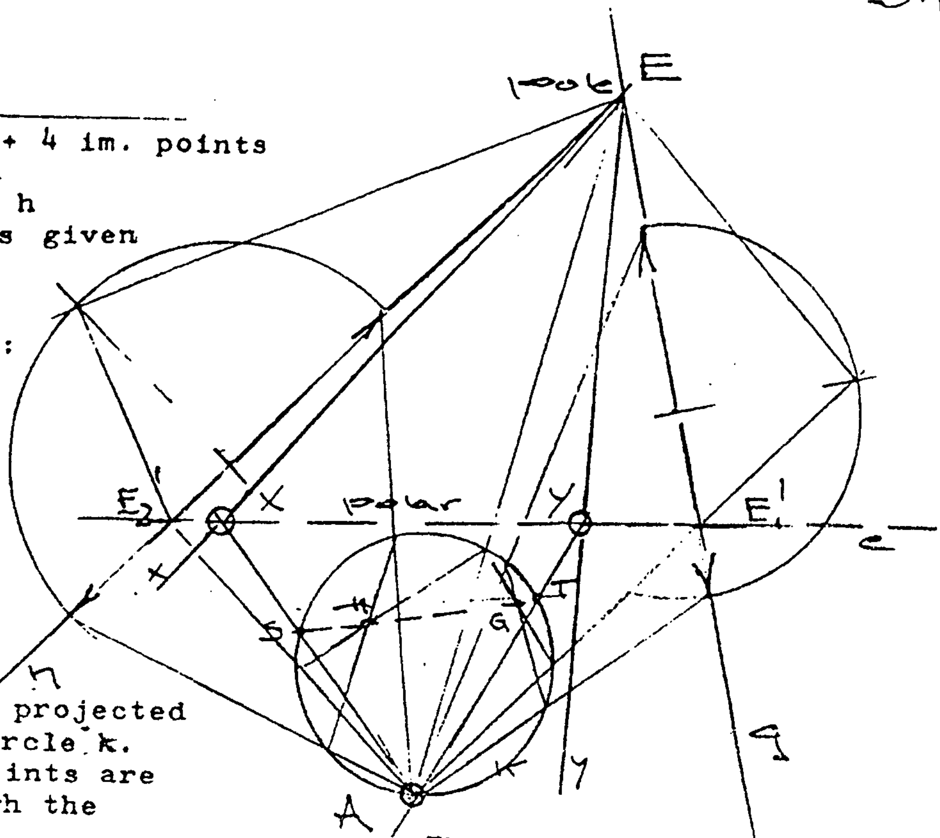
As b is at infinity, the curve must be a parabola. b' is the vertex tangent. The polar line is the directrix.

Problem

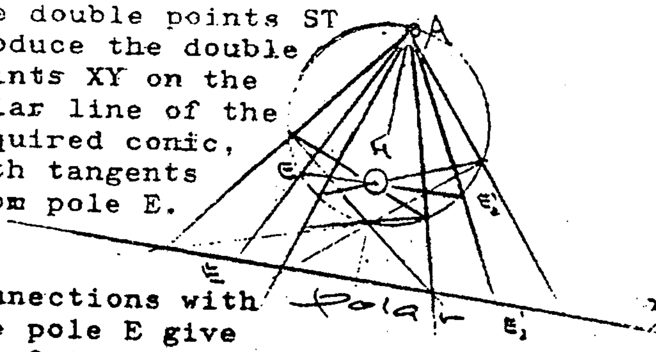
given : 1 real + 4 im. points of a conic, and the lines g and h with involutions given by arrows.

Find the conic.

Conjugate pairs:
E E_1' E E_2'.
The line $E_1'E_2'$ is polar to E.

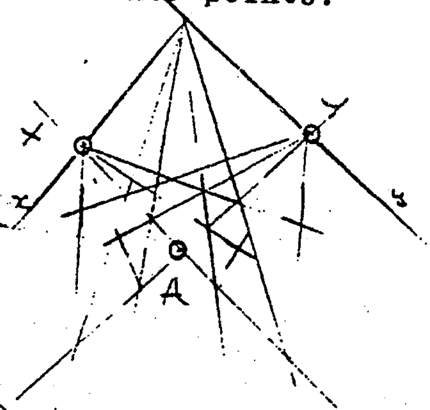


Involutions are projected from A to the circle k. Corresponding points are connected through the centres H and G. The double points ST produce the double points XY on the polar line of the required conic, with tangents from pole E.



Connections with the pole E give the 2 tangents in these points (X Y).

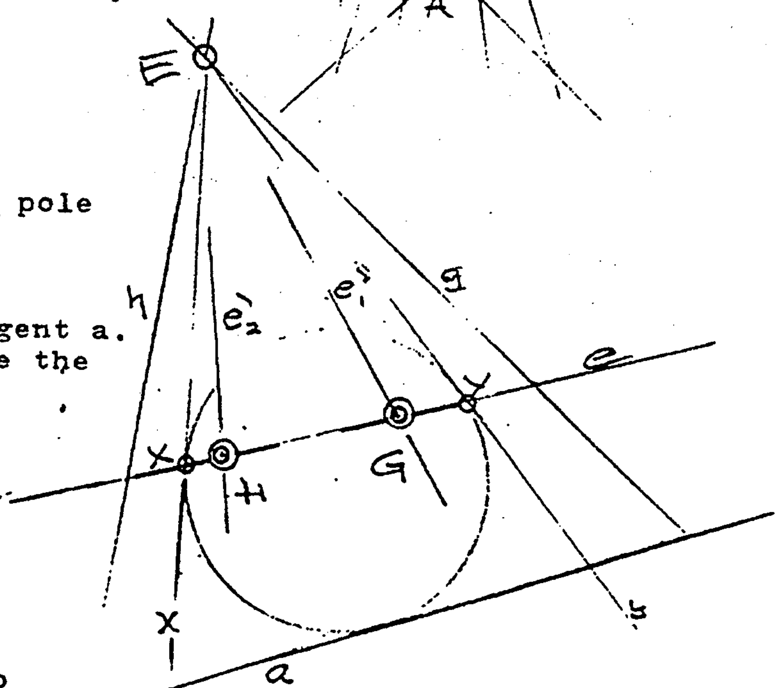
Three points with two tangents: construction of further points.

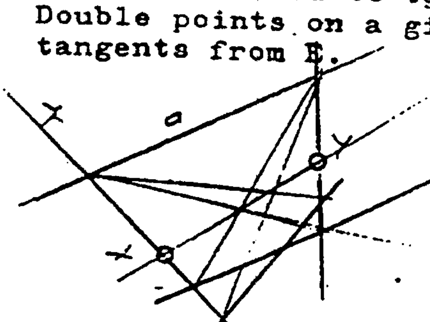


Polar problem

given : 1 real + 4 im.tangents.

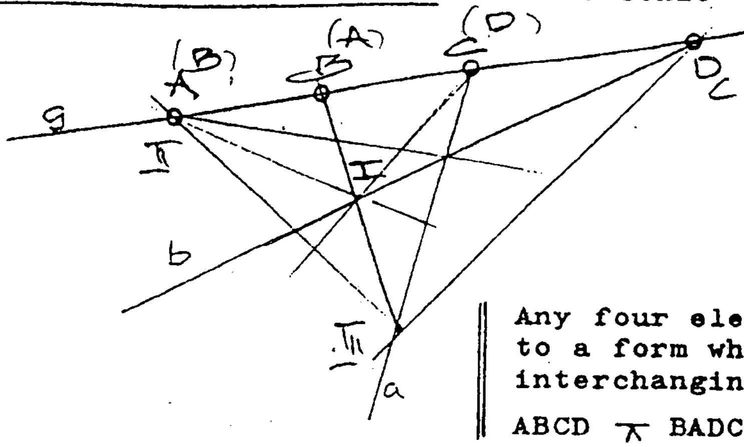
e_1' e_2' intersect at the pole to the line H-G. Projections of the involutions in H and G are transferred to tangent a. Double points on a give the tangents from E.





Three tangents with two points:construction of further tangents.

QUADRANGLE on the conic and TRANSVERSAL

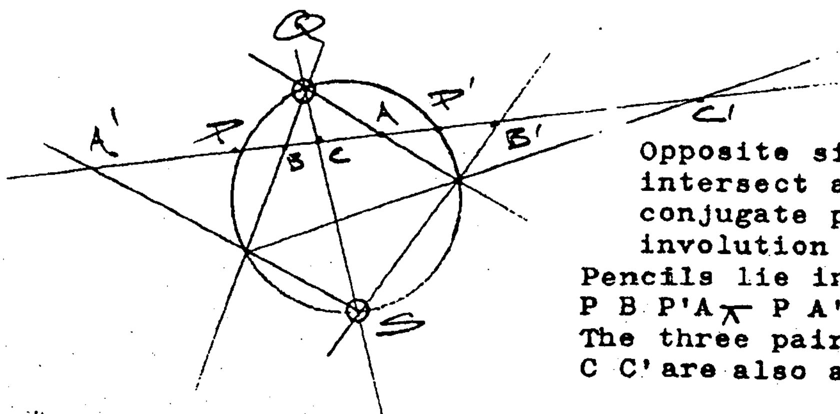


ABCD are projected through I to a
" II to b
" III to g

A B C D ⊼ B A D C
Elements have interchanged in pairs.

Any four elements are projective to a form which is derived from interchanging elements in pairs.

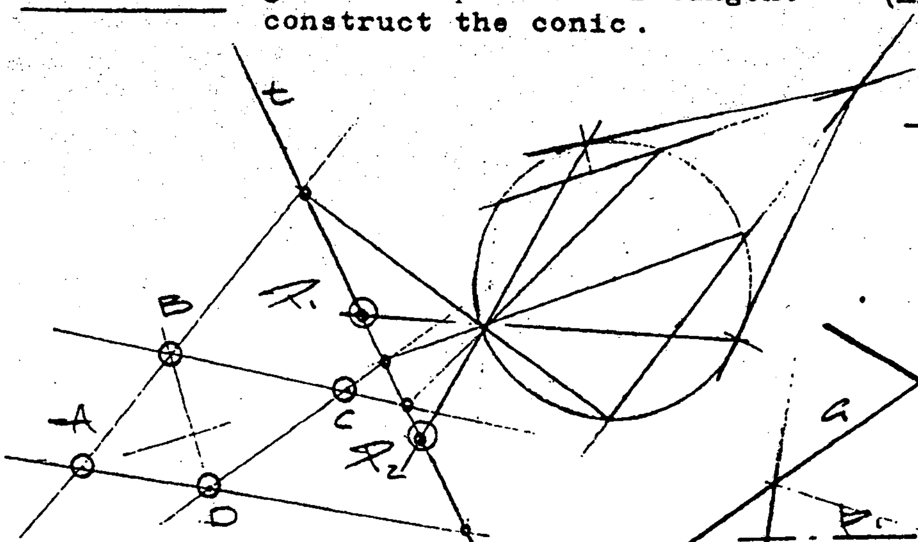
ABCD ⊼ BADC ⊼ CDAB ⊼ DCBA



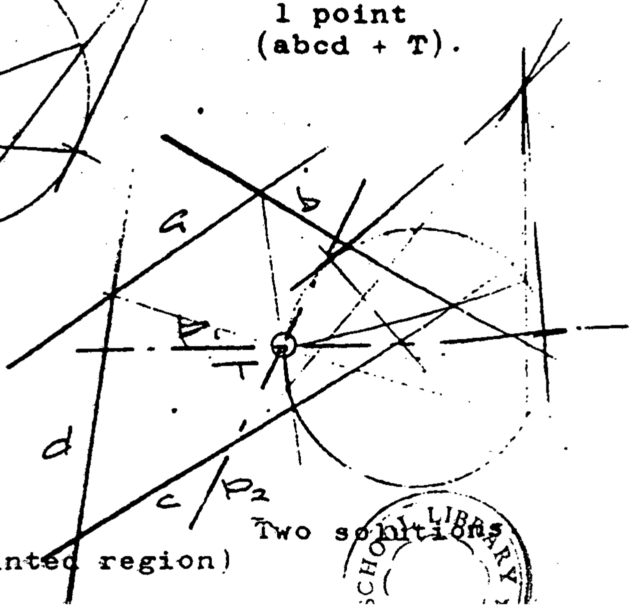
Opposite sides of a quadrangle intersect a transversal in conjugate points of an involution (page 41).
Pencils lie in S and Q. Therefore
P B P'A ⊼ P A'P'B' ⊼ P'B'P A'.
The three pairs are in involution-
C C' are also a conjugate pair.

Conics through four points (pencil of conics) and the sides of the quadrangle of the same four points intersect the transversal in pairs of conjugate points of an involution.

Problem : given 4 points + 1 tangent (ABCD + t), construct the conic.



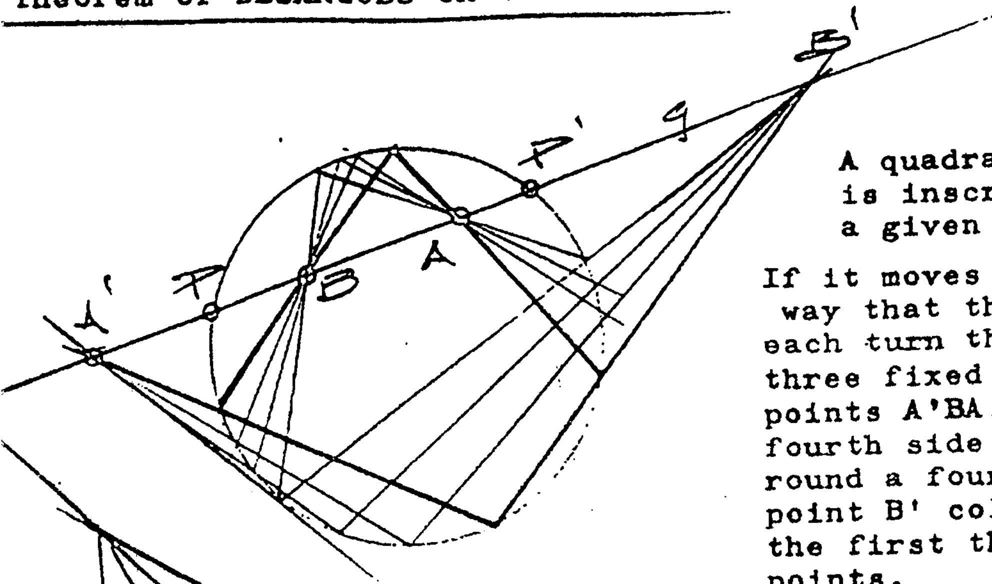
Polar problem: given: (below) 4 tangents and 1 point (abcd + T).



Two solutions-The auxiliary circle gives two double points of the involution on the tangent t. These double points are a touching point each to two different conics.
(No solutions if t in three-pointed region)

Two solutions

Theorem of DESARGUES on the CONIC

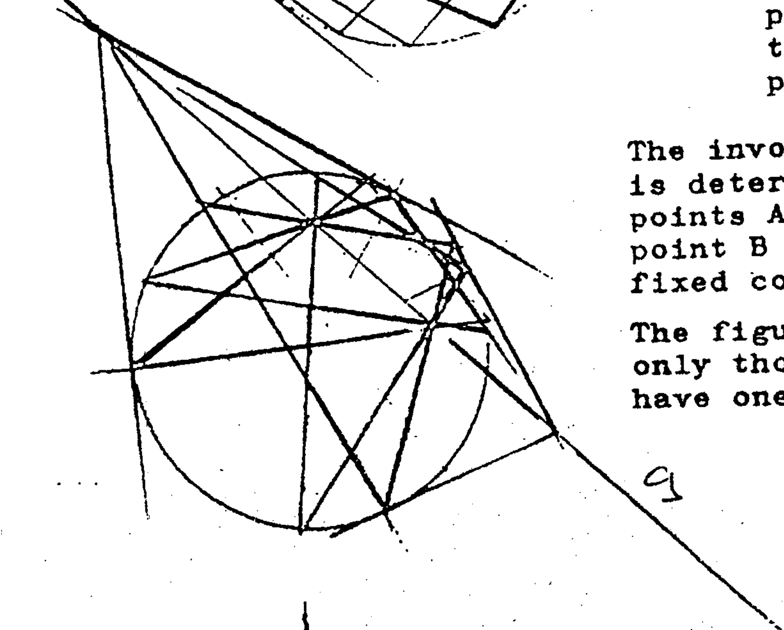


A quadrangle is inscribed in a given conic.

If it moves in such a way that three sides each turn through one of three fixed collinear points A'BA, the fourth side will turn round a fourth fixed point B' collinear with the first three given points.

The involution on the line g is determined by the four points A A'P P'. To a fixed point B corresponds a fixed conjugate point B'.

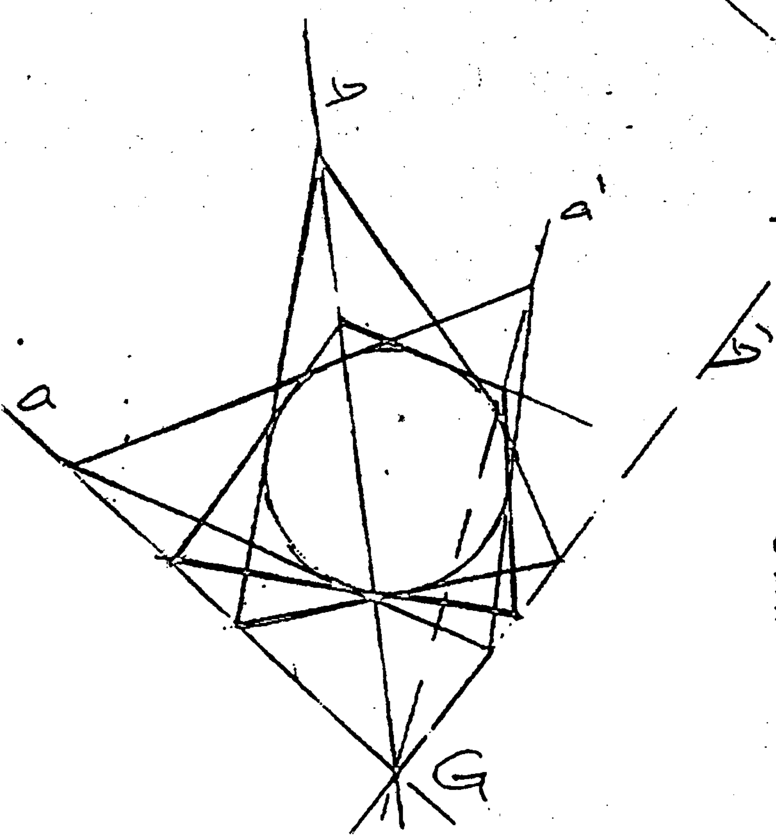
The figure on the left shows only those quadrangles which have one side as a tangent.



A quadrilateral-(Polar Configuration)

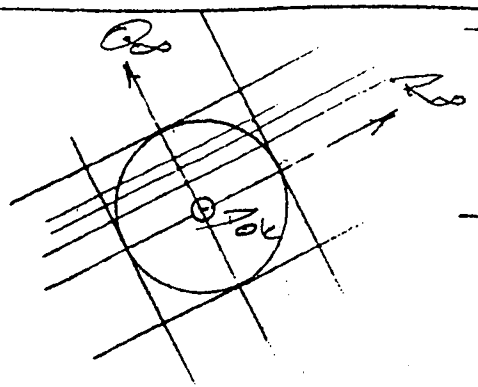
Four tangents to a conic.

The intersections of the tangents glide on four lines through a point.



Construct any inscribed polygon with an even number of corners, sides rotating round collinear fixed points.

CENTRES, DIAMETERS and AXES of conic curves

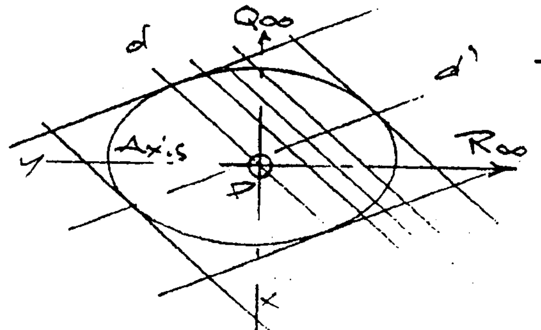


—C E N T R E—
is the pole to the line
at infinity.
P Q R is a polar triangle
with one side at infinity.

—D I A M E T E R S—
are lines through the centre.

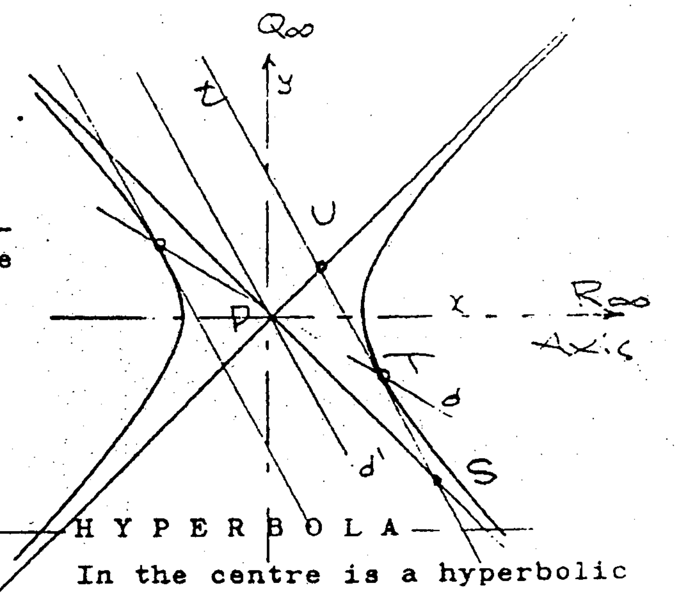
The ray involution in the
centre of a circle is a
right-angled involution.

Tangents at the intersections of a diameter with the conic are parallel to the conjugate diameter. The centre is the middle of the diameter.



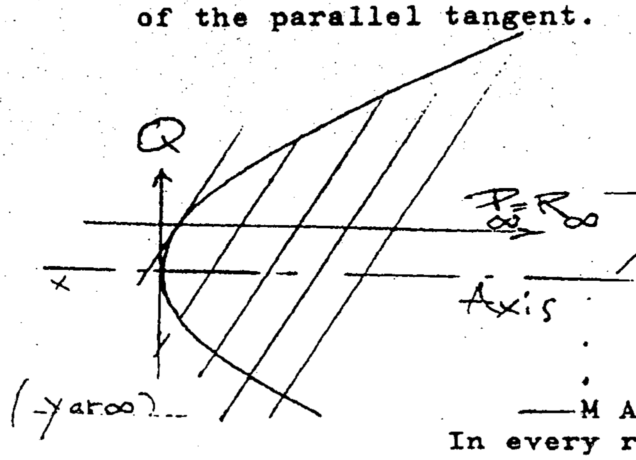
— E L L I P S E—

Conjugate diameters form
an elliptic involution
in the centre.



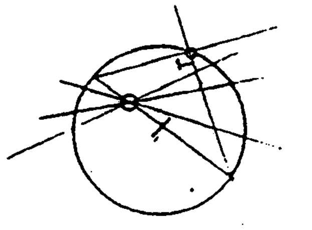
—P A R A B O L A—

The centre is at
infinity, diameters are
parallel, chords are inter-
sected in the middle by the
line parallel to the axis
through the touching point
of the parallel tangent.

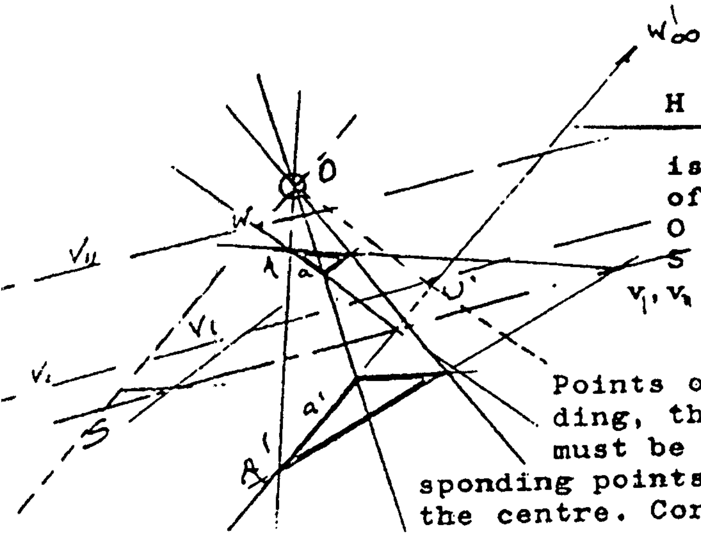


—H Y P E R B O L A—
In the centre is a hyperbolic
involution. Asymptotes are
double rays. Conj. diameters dd'
are harm. with the asymptotes.
Segment UT=TS on any tangent t.

—M A I N A X E S of a C O N I C—
In every ray involution exists one pair of
conjugate rays which are at right angles
to each other.
Conjugate diameters of a conic curve at
right angles are called the MAIN AXES
of the conic (x and y).



HOMOLOGY

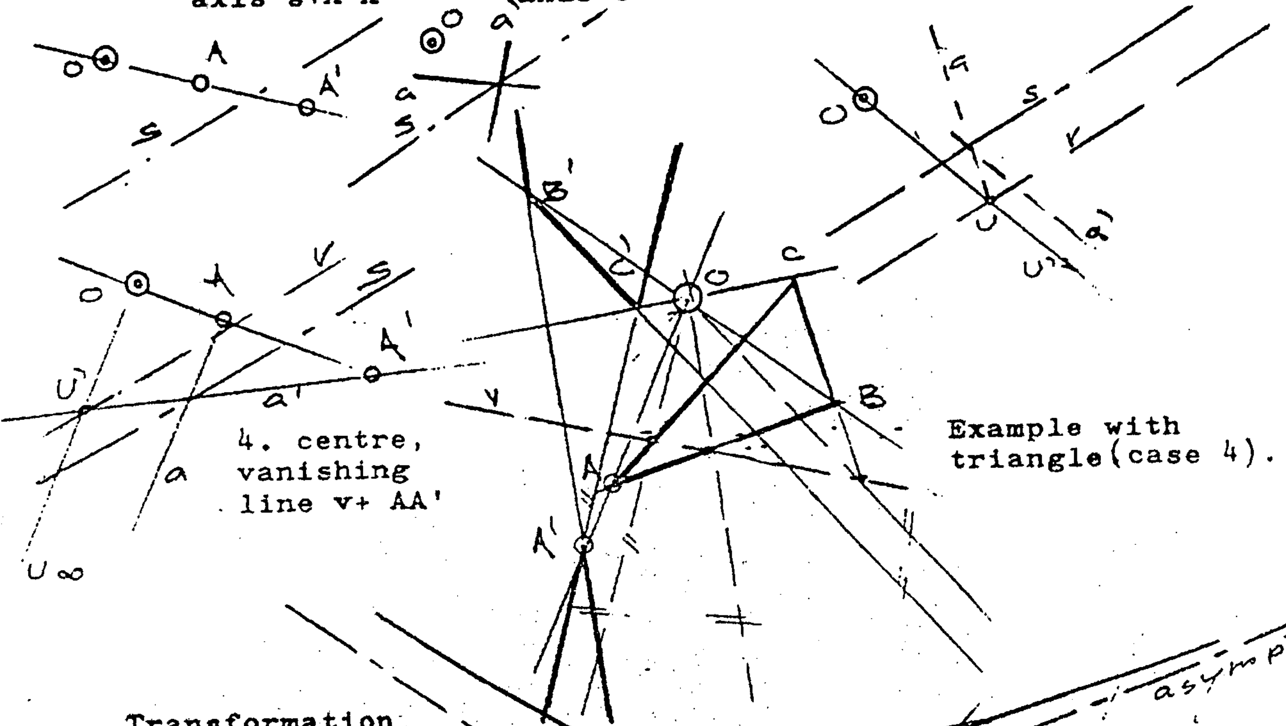


is a perspective transformation of the plane in itself.
O : centre of homology
S ; axis of homology
v_1, v_1' : vanishing lines, corresponding to the line at infinity.

Points on the axis are self-corresponding, therefore the vanishing lines must be parallel to the axis. Corresponding points (e.g. AA') are collinear with the centre. Corresponding lines (e.g. aa') intersect on the axis.

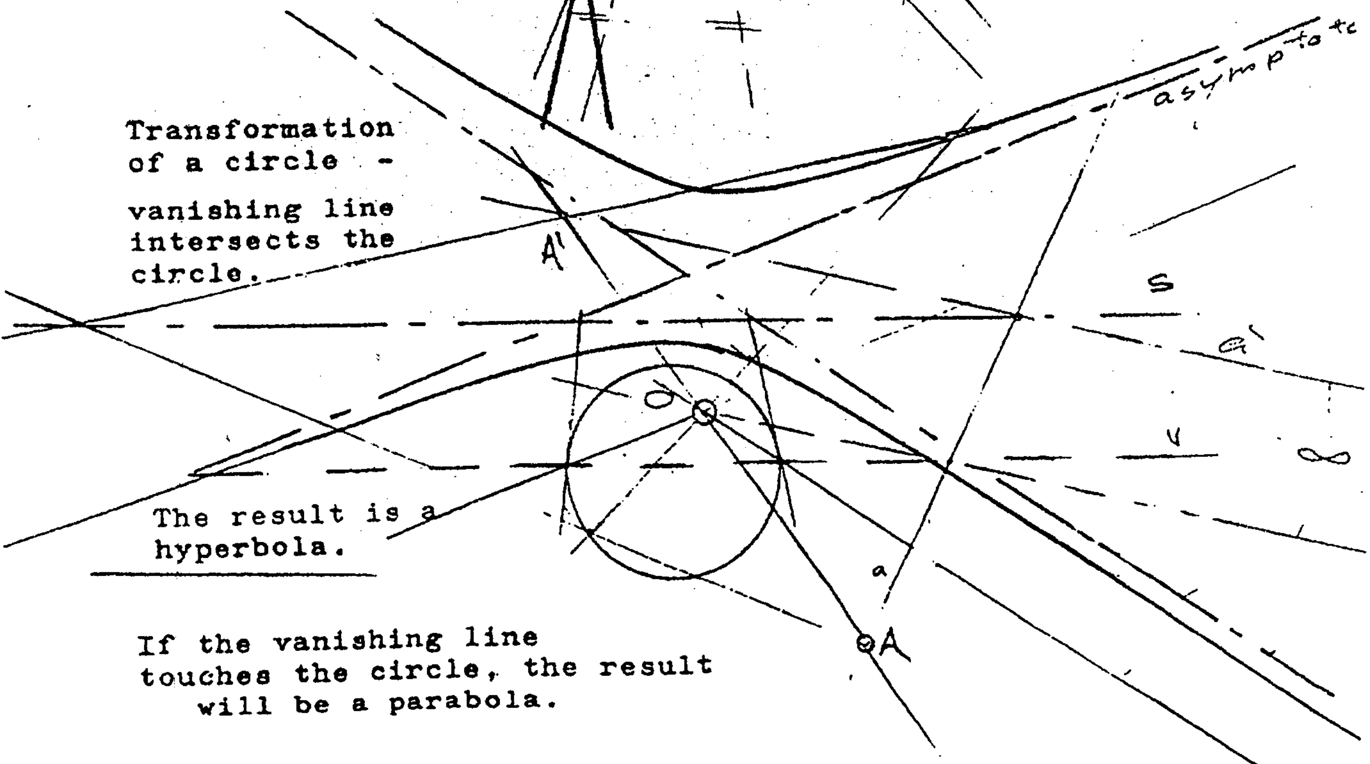
Homology is determined with:

1. centre, axis s+A A'
2. centre, axis s+a a'
3. centre, axis + vanishing line v



4. centre, vanishing line v+ AA'

Example with triangle (case 4).

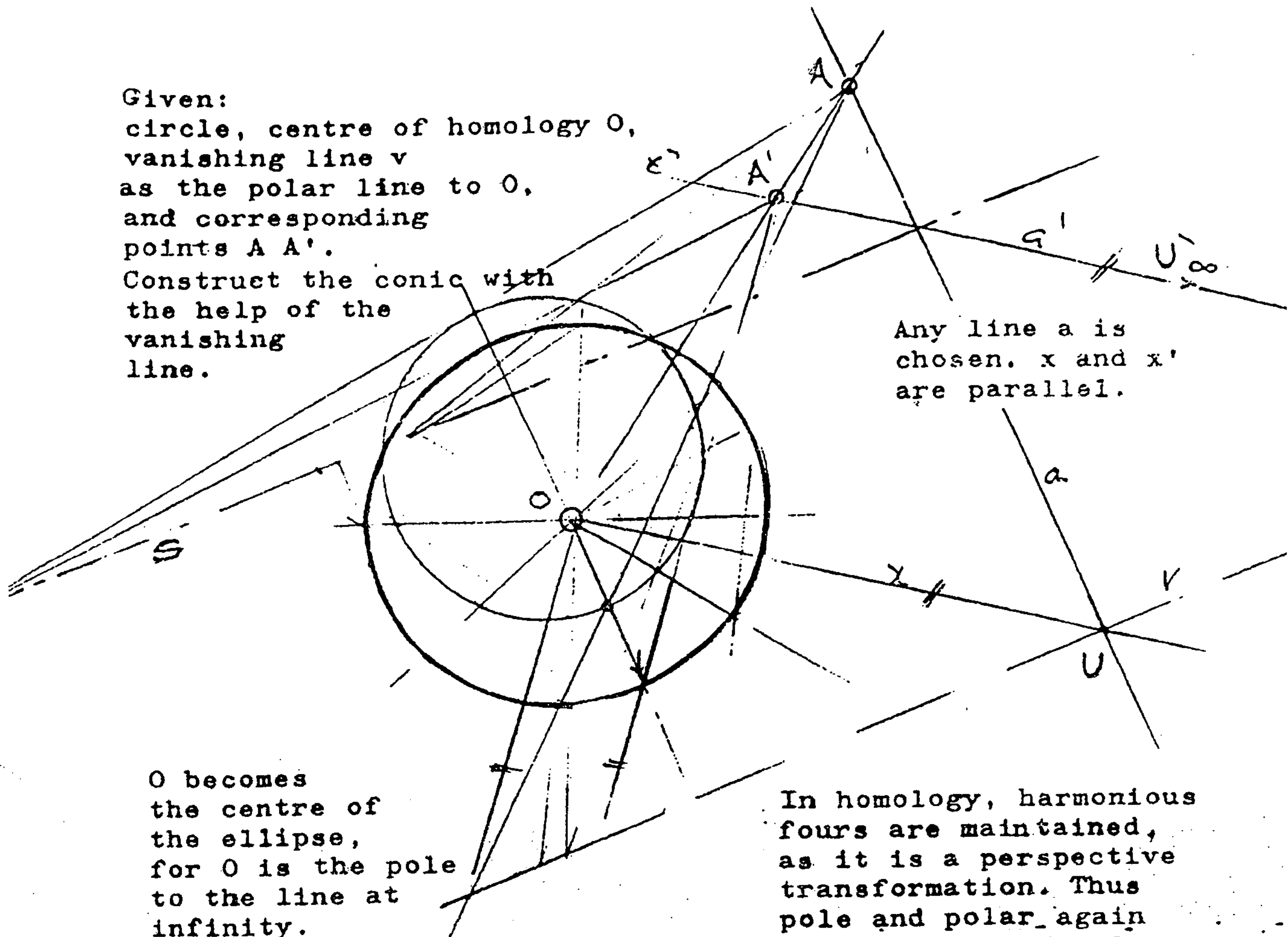


Transformation of a circle - vanishing line intersects the circle.

The result is a hyperbola.

If the vanishing line touches the circle, the result will be a parabola.

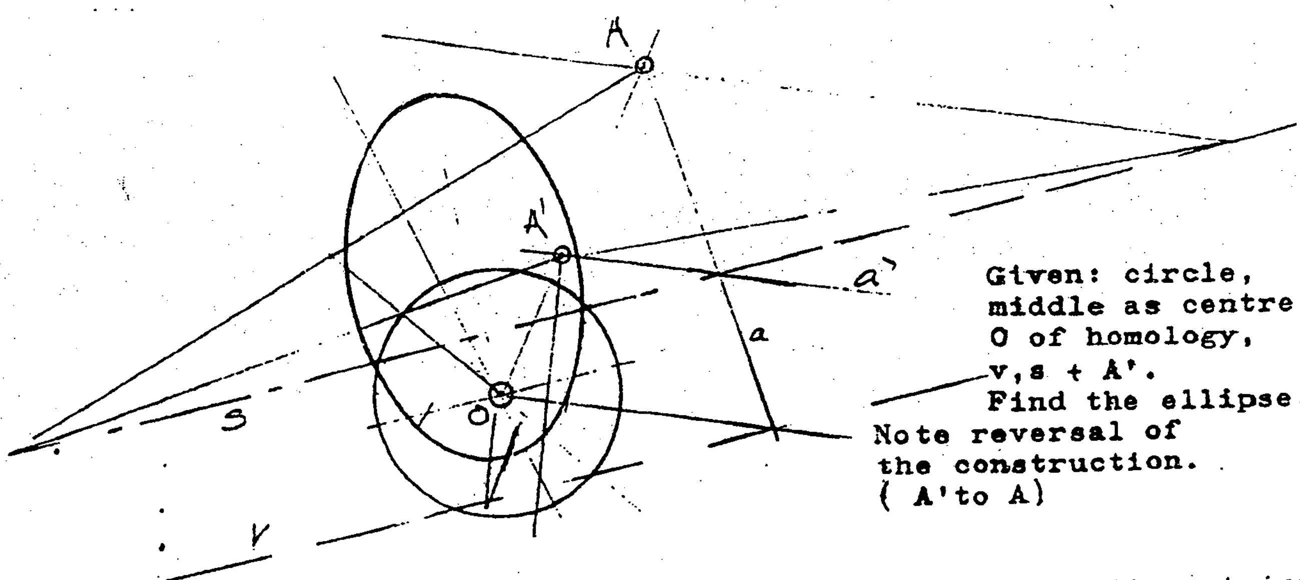
Given:
circle, centre of homology O,
vanishing line v
as the polar line to O,
and corresponding
points A A'.
Construct the conic with
the help of the
vanishing
line.



Any line a is chosen. x and x' are parallel.

O becomes
the centre of
the ellipse,
for O is the pole
to the line at
infinity.

In homology, harmonious fours are maintained, as it is a perspective transformation. Thus pole and polar again become pole and polar.



Given: circle,
middle as centre
O of homology,
v, s + A'.
Find the ellipse.
Note reversal of
the construction.
(A'to A)

The FOCUS POINT OF A CONIC
has a right-angled
ray involution.

The polar to O is the line at infi[n]
The ray involution is right-angled
at the centre O.
O remains fixed , therefore
the involution in O also remains
fixed for the ellipse,
and is the focus of the ellipse.

COLLINEATION

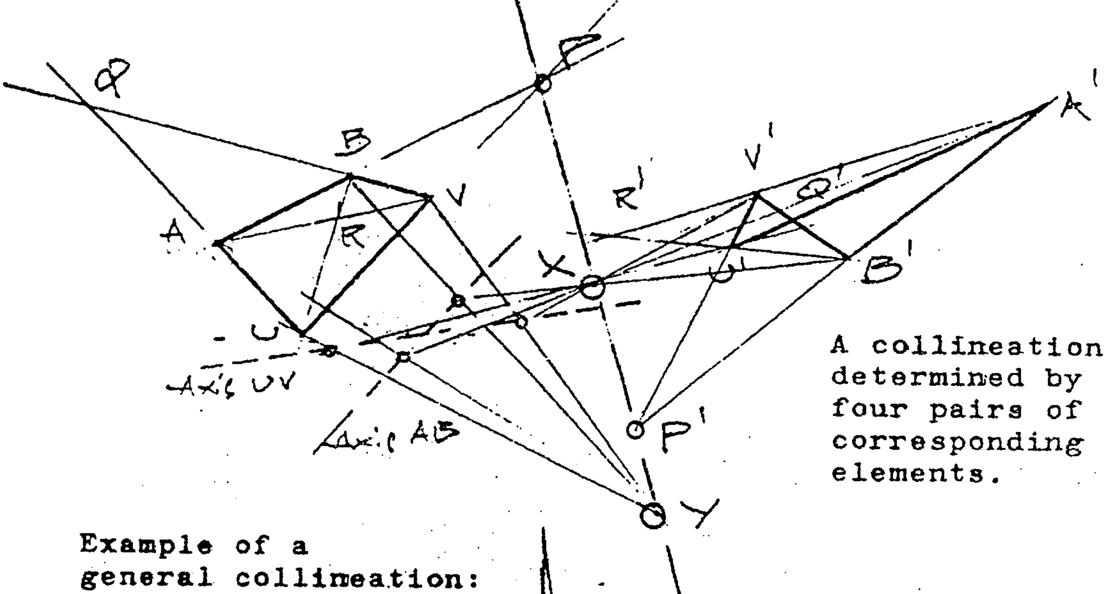
is a one-to-one correspondence between two planes. A point in one plane corresponds to one and only one point in the other plane. A line corresponds to one and only one line. The order is maintained. Inside elements correspond to inside elements.
If both fields are in the same plane it is called a transformation of the plane in itself.

Two corresponding quadrangles or quadrilaterals determine a collineation.

Corresponding formations of the first degree are projectively related. Therefore ranges and pencils of the second degree are also projectively related.

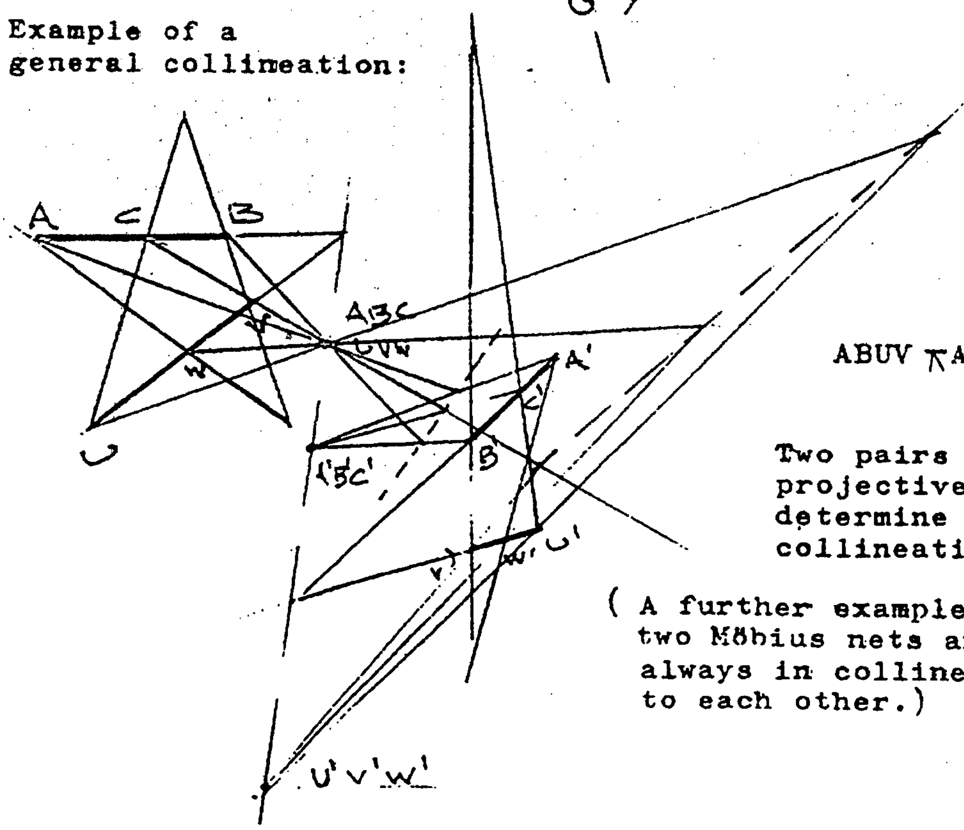
A B P $\barwedge$ A'B'P'
U V P $\barwedge$ U'V'P'

choose on PP'
two points X Y



A collineation is determined by four pairs of corresponding elements.

Example of a general collineation:



ABUV $\barwedge$ A'B'U'V'

Two pairs of projective ranges determine a collineation.

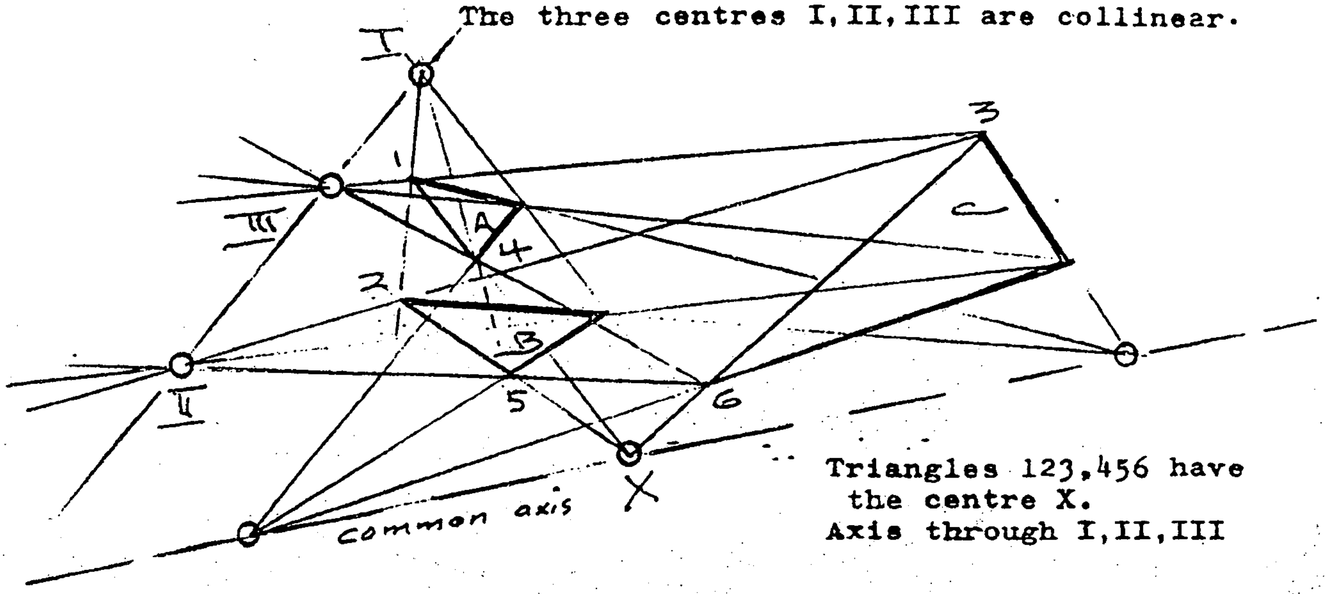
(A further example is that two Möbius nets are always in collineation to each other.)

CENTRAL COLLINEATION

The lines through the centre (fixed point) are individually self-corresponding.
It follows from Desargues' theorem that a line exists (the axis) whose points correspond to themselves.
(This theorem is self-polar ; see page 9.)
These relations are thus perspective.

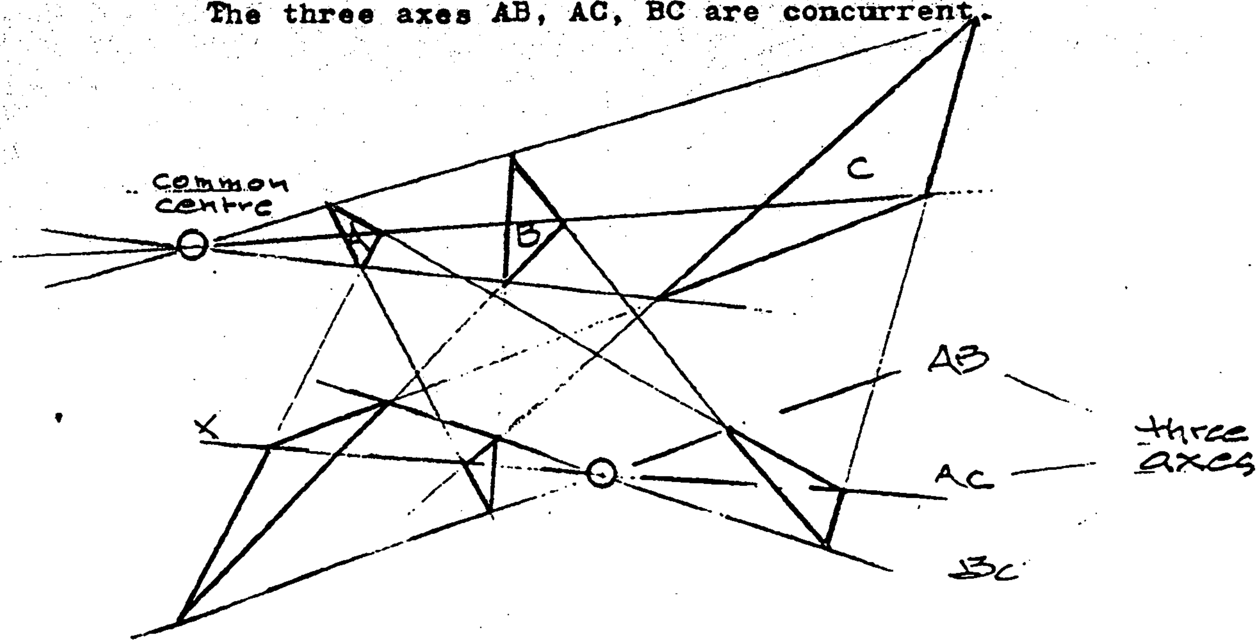
Two central collineations in combination (A to B and B to C) :

1. With common axis :
   produce a third collineation (A to C).
   The three centres I, II, III are collinear.



Triangles 123,456 have the centre X.
Axis through I,II,III

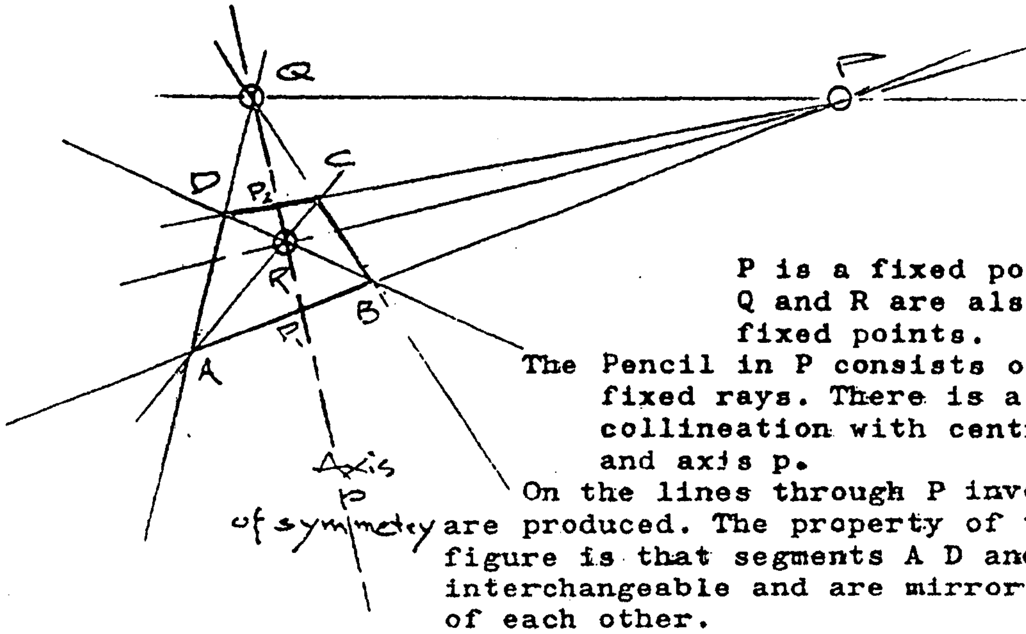
2. With common centre :
   produce a third collineation (A to C).
   The three axes AB, AC, BC are concurrent.



C E N T R A L and A X I A L S Y M M E T R I E S

are harmonious mirror transformations
and belong to central collineations.

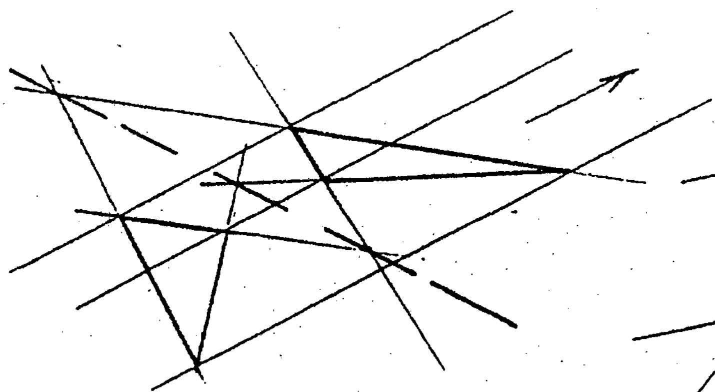
All pairs of corresponding points and all pairs of corresponding lines are harmoniously separated by the centre P and the axis p.

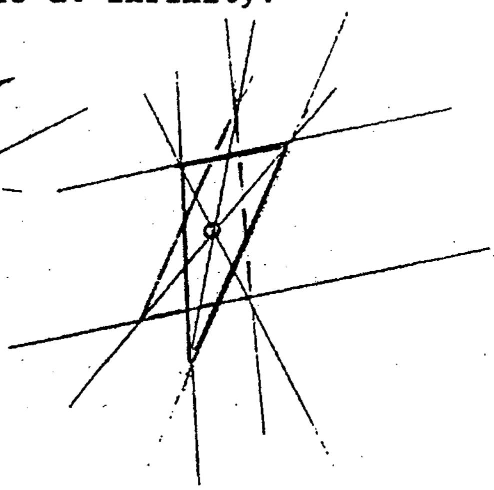


P is a fixed point.
Q and R are also fixed points.
The Pencil in P consists of fixed rays. There is a collineation with centre P and axis p.
On the lines through P involutions are produced. The property of this figure is that segments A D and B C are interchangeable and are mirror pictures of each other.

AXIAL SYMMETRY -
centre at infinity.

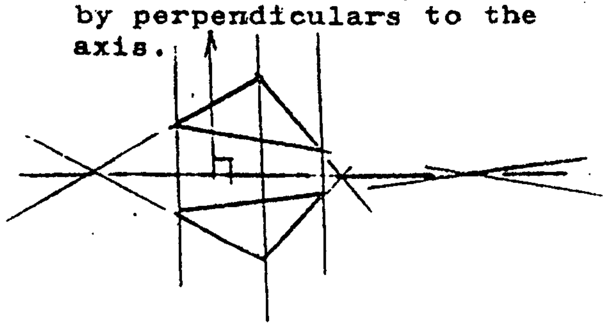
CENTRAL SYMMETRY -
axis at infinity.





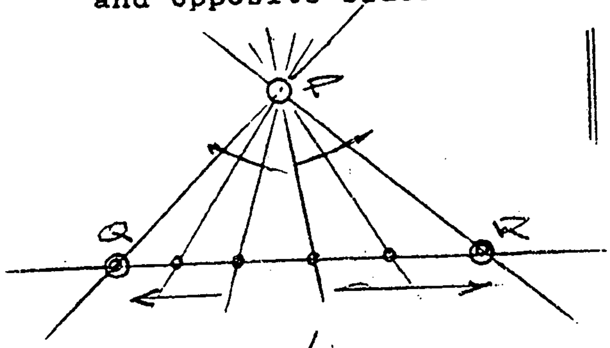
Harmonious mirror transformations are collineations in involution., Correspondence is interchangeable.
(On each line through the centre are 2 double points.)
Every involution of a conic curve determines a harmonious mirror transformation. The conic is transformed into itself.

COMMON MIRROR TRANSFORMATION —
centre at infinity reached by perpendiculars to the axis.

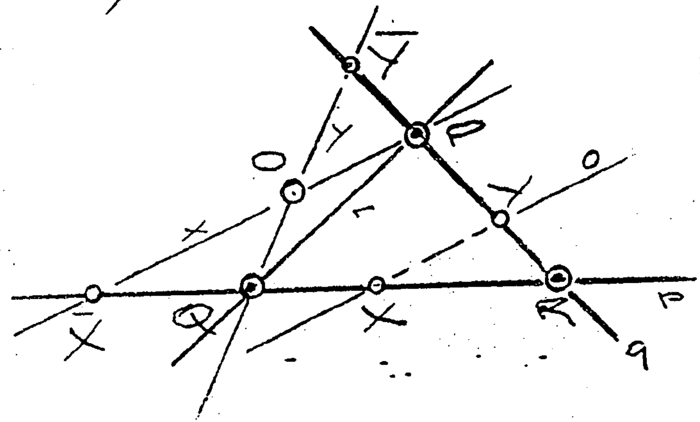


P O L A R   S Y S T E M S

A polar system is a reciprocal correspondence, in a plane, in itself. Point corresponds to line, line corresponds to point. There exists a triangle of which points and opposite sides are interchangeably correspondent.



Two points are called conjugate if they are on each other's polar line. Two conjugate lines go through each other's pole. Conjugate points on a line and conjugate lines through a point each form an involution. (Look at the polar triangle on page 32.)



P Q R : polar triangle

O o : pole and polar

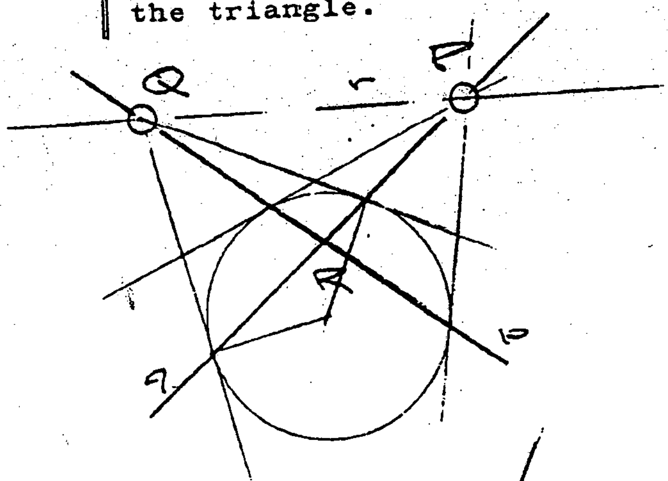
X $\overline{X}$, Y $\overline{Y}$: conjugate pairs of points.

X x, Y y : corresponding pairs of pole and polar (the polar to X goes through P and O).

P Q R O -- p q r o

This quadrangle and quadrilateral determine a reciprocity in the plane in itself.

A polar system is determined by a polar triangle (P Q R - pqr) plus one pair of corresponding elements (O o). O is not on a side and o is not through a corner of the triangle.

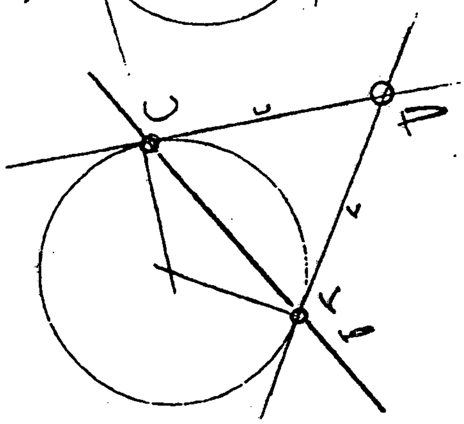


Choose P, not on its polar p. Q on p, not on its polar q. QPR is thus a polar triangle.

PQ QR RP are each a pair of conjugate points as they are on each other's polar lines.

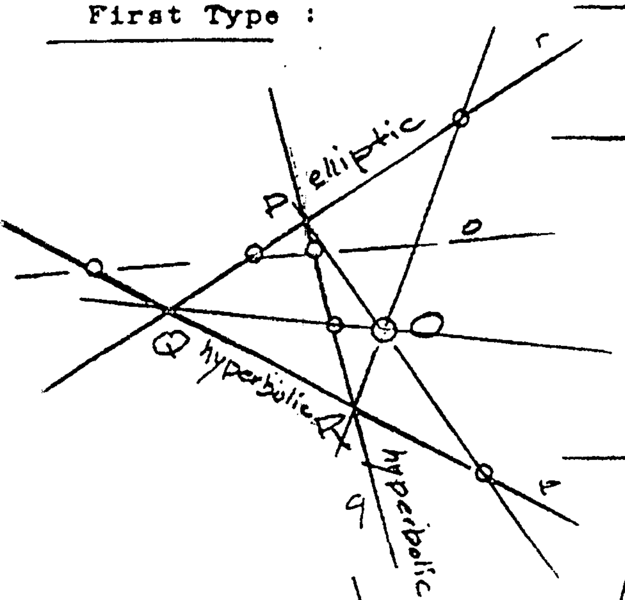
Equally are pq qr rp each a pair of conjugate lines.

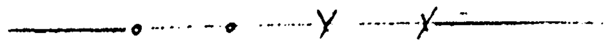
If U is on u and V on v : u and v are tangents to the conic.
This tangential triangle is not a polar triangle.



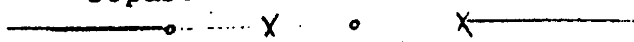
THERE ARE TWO POLAR SYSTEMS essentially different from each other.

First Type :





Hyperbolic involution.
Corresponding pairs do not separate each other.



Elliptic involution.
Corresponding pairs separate each other.

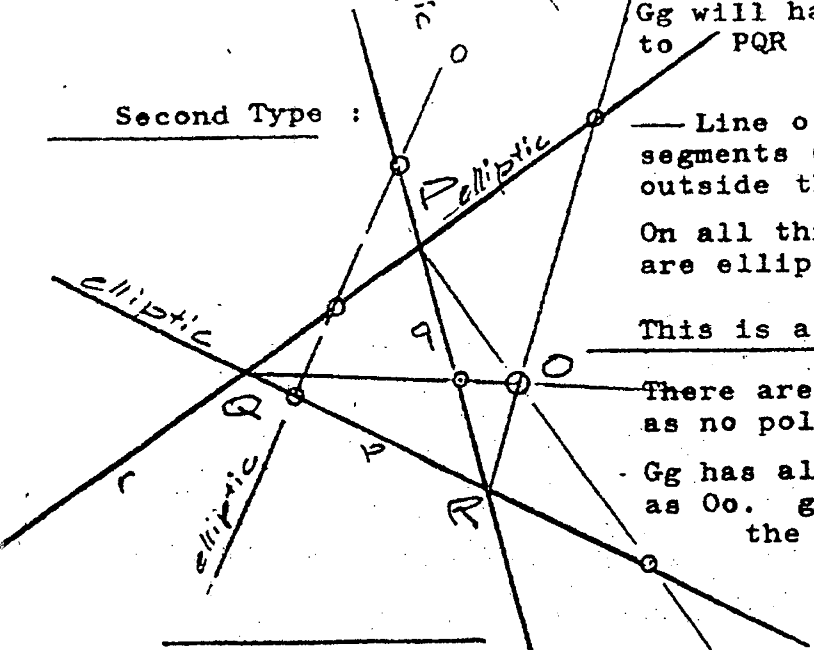
PQR Oo determine the system.

On p : hyperbolic involution.
On q : hyperbolic involution.
On r : elliptic involution.

This is a mixed system.

A polar g to any point G is constructed with the conjugate points.
Gg will have the same relation to PQR as Oo.

Second Type :



— Line o intersects all three segments of the triangle outside the section containing O.

On all three sides (pqr) there are elliptic involutions.

This is a uniform system.

There are no real double points as no pole can be on its polar.

Gg has always the same relation as Oo. g never intersects the section containing G.

All involutions in this second case are elliptic.
The corresponding conic curve is an imaginary one.
(Christian von Staudt 1847)

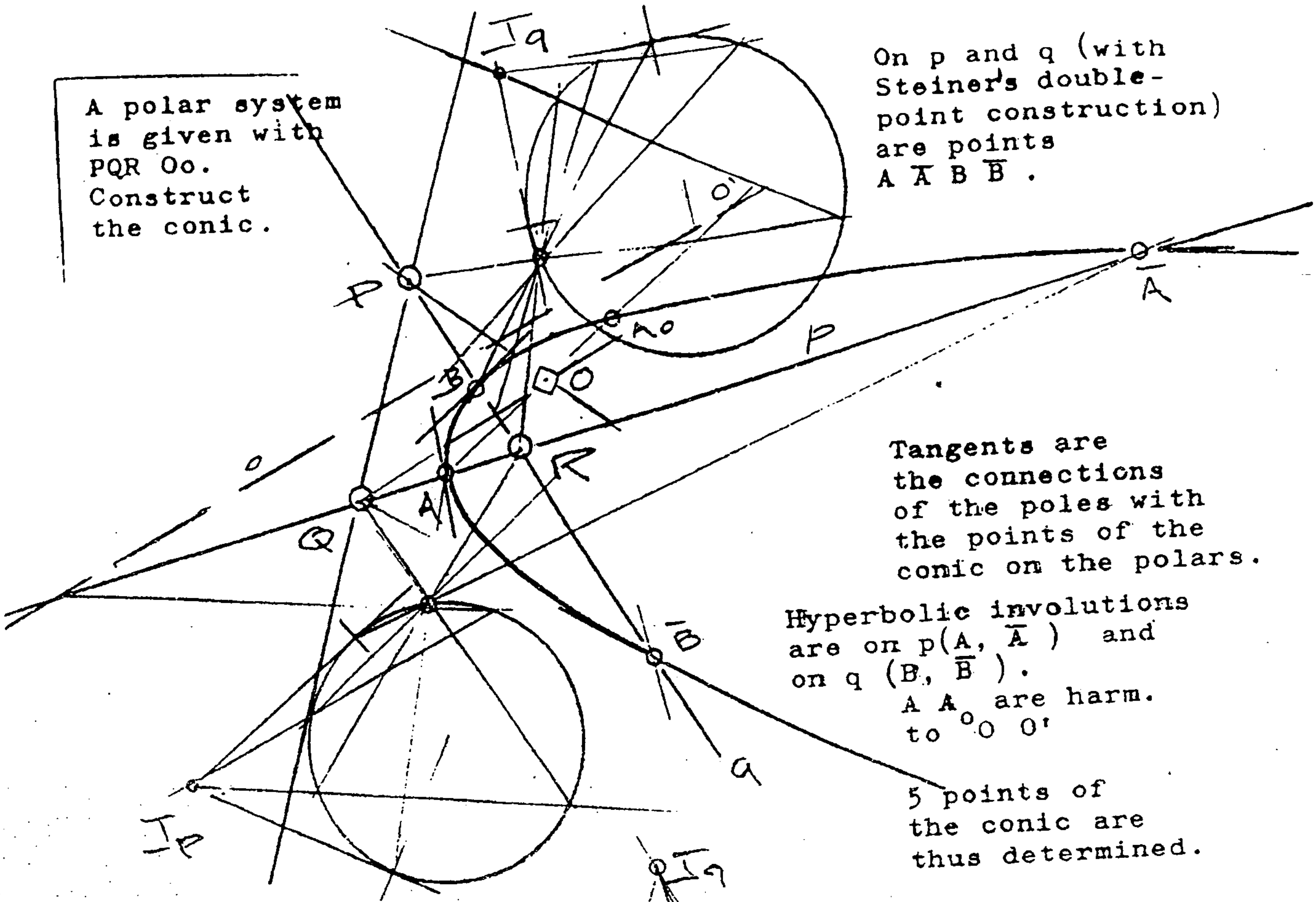
If there is one polar triangle with mixed involutions, all polar triangles of the system are also mixed (first type).
If there is one polar triangle with uniform involutions, then all are uniform (second type).

Every conic curve defines, in its plane, a polar system and every polar system determines a conic.
The conic curve consists of :

all real or imaginary double points on the lines produced by the involutions or –	all real or imaginary double rays through the points produced by the involutions.

A polar system is given with PQR Oo. Construct the conic.

On p and q (with Steiner's double-point construction) are points A $\bar{A}$ B $\bar{B}$.



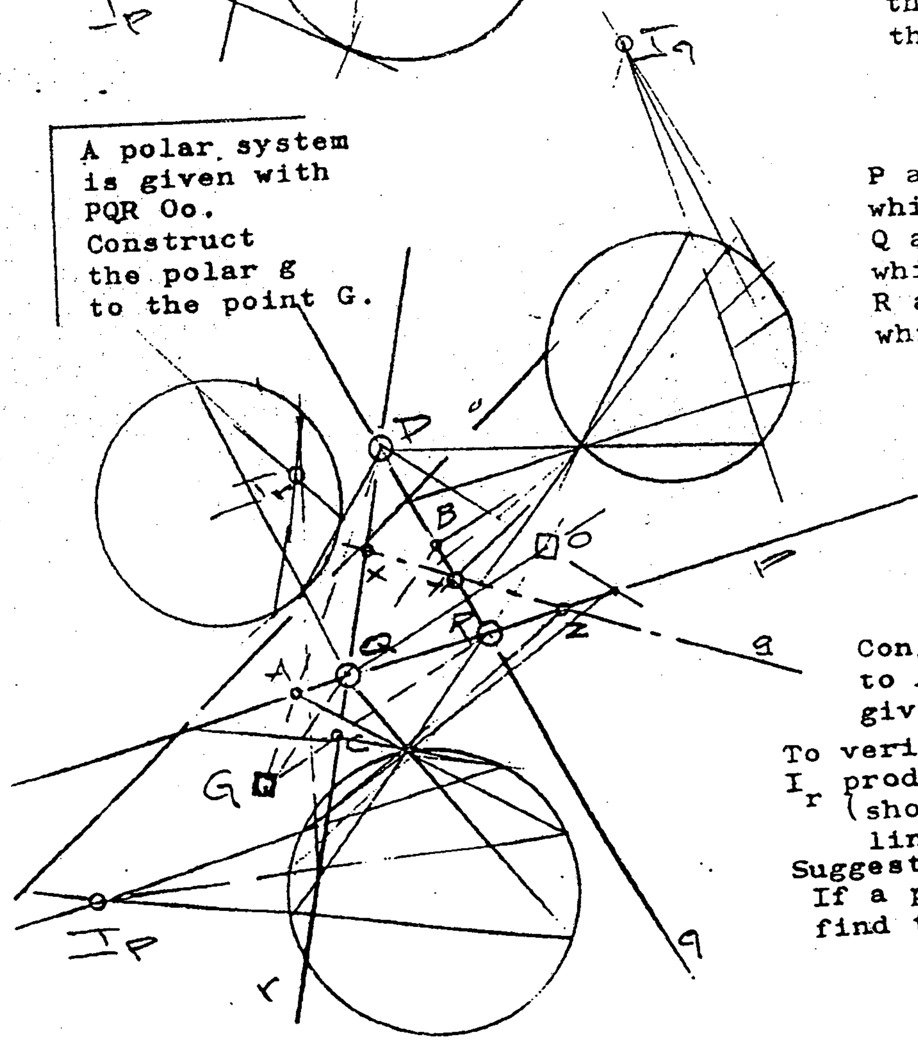
Tangents are the connections of the poles with the points of the conic on the polars.

Hyperbolic involutions are on p(A, $\bar{A}$) and on q (B, $\bar{B}$).
A A_o are harm. to O_o O'

5 points of the conic are thus determined.

A polar system is given with PQR Oo. Construct the polar g to the point G.

P and G lie on a line which intersects p at A.
Q and G lie on a line which intersects q at B.
R and G lie on a line which intersects r at C.

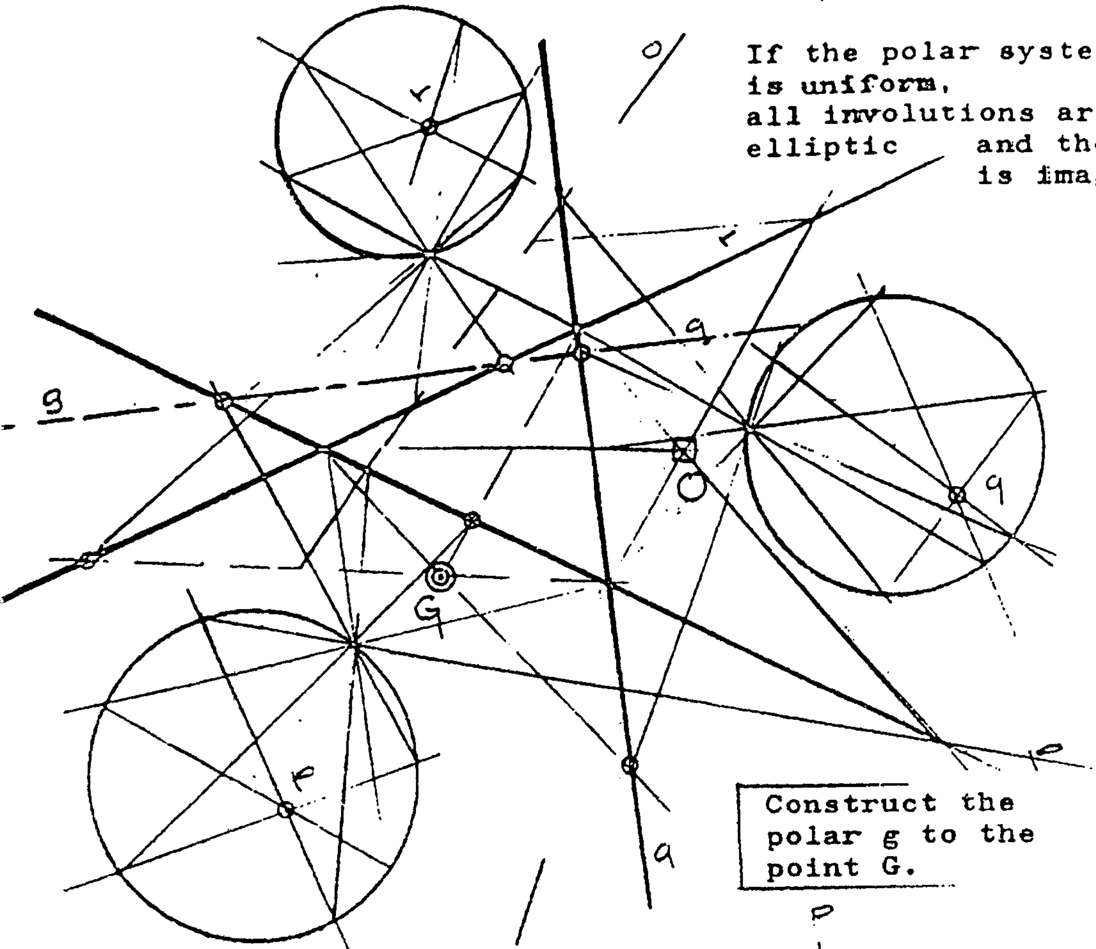


Conjugate points (XYZ) to ABC on pqr give polar g to point G.

To verify :
I_r produces point X on r (should also be on line g with Y Z).

Suggestion :
If a polar line is given, find the pole.

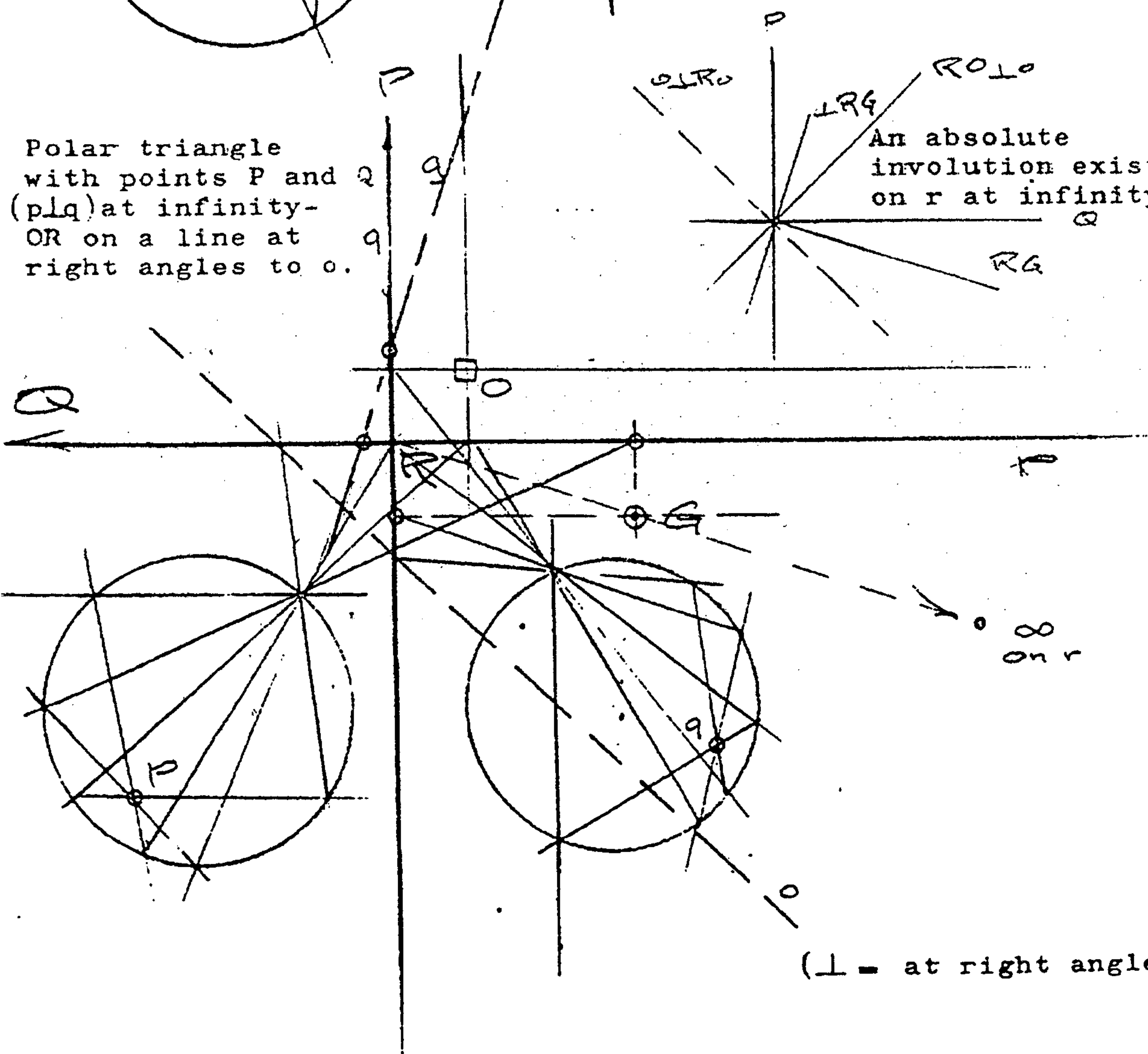
If the polar system is uniform, all involutions are elliptic and the conic is imaginary.



Construct the polar g to the point G.

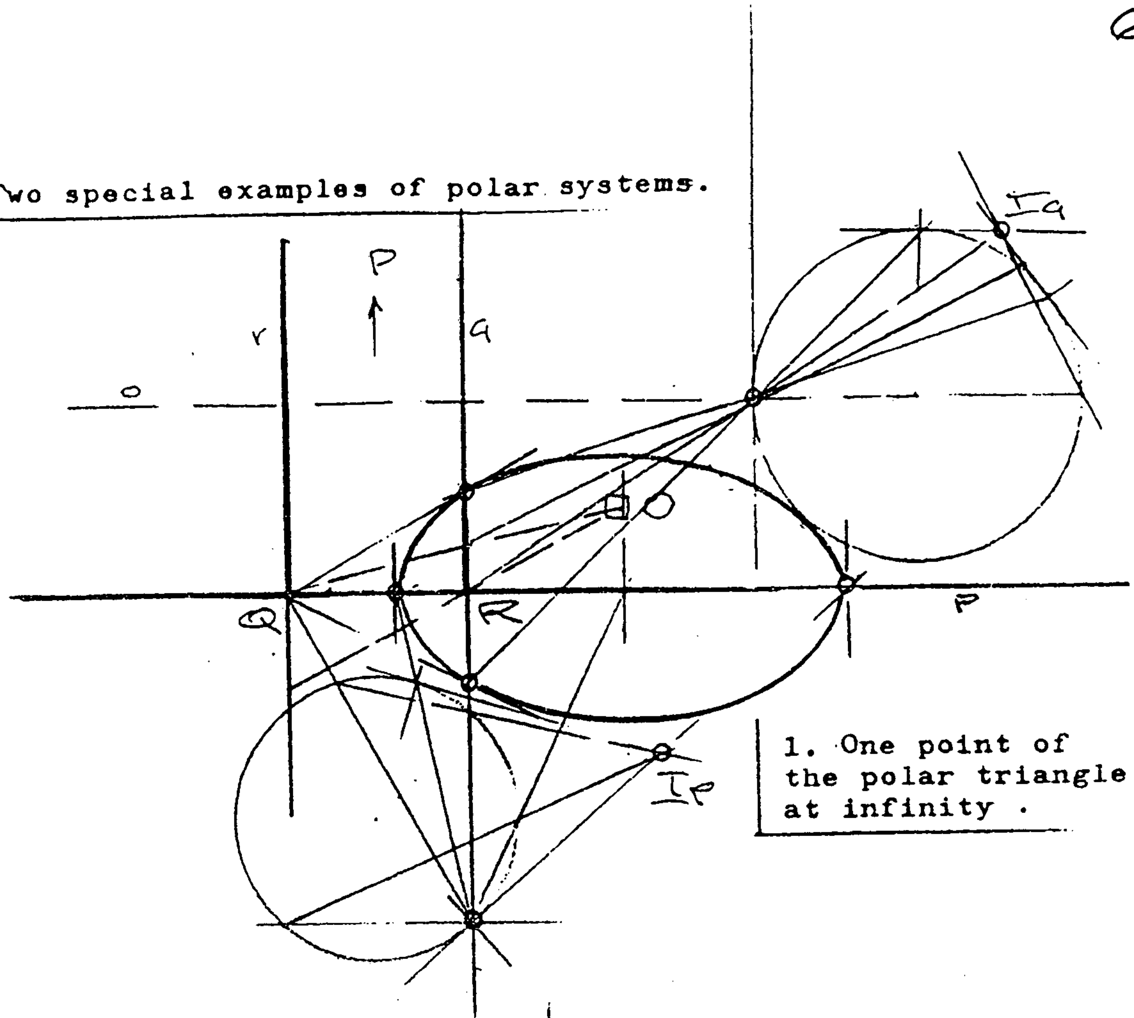
Polar triangle with points P and Q (p⊥q) at infinity- OR on a line at right angles to o.

An absolute involution exists on r at infinity.



(⊥ = at right angles)

Two special examples of polar systems.

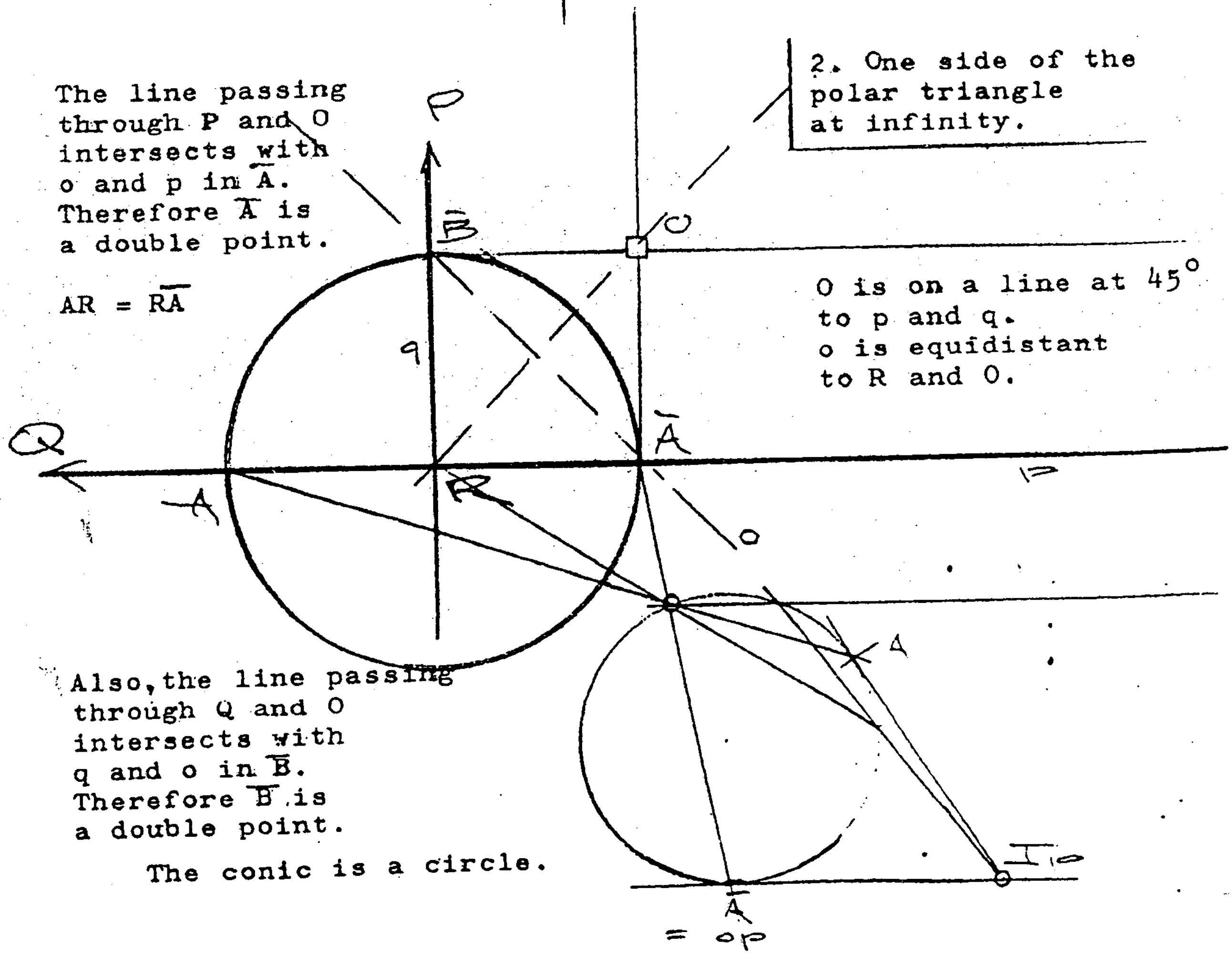


1. One point of the polar triangle at infinity.

2. One side of the polar triangle at infinity.

The line passing through P and O intersects with o and p in $\bar{A}$. Therefore $\bar{A}$ is a double point.

$AR = \overline{RA}$



O is on a line at 45° to p and q.
o is equidistant to R and O.

Also, the line passing through Q and O intersects with q and o in $\bar{B}$. Therefore $\bar{B}$ is a double point.

The conic is a circle.

S U P P L E M E N T to:

I N T R O D U C T I O N to PROJECTIVE GEOMETRY

w o r k i n g n o t e s f o r s t u d e n t s

Angelo Andes Rovida 1980

Edited by Norman Davidson

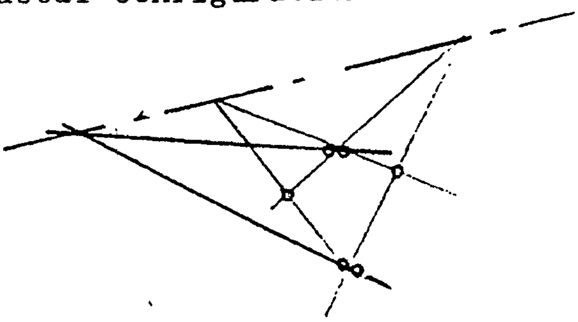
Page

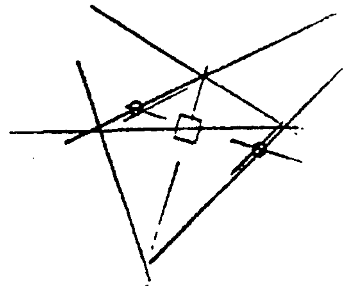
From Th. Reye -
Geometrie der Lage

POLAR THEORY with respect to CONICS

Four points on a conic with two tangents produce the Pascal configuration.

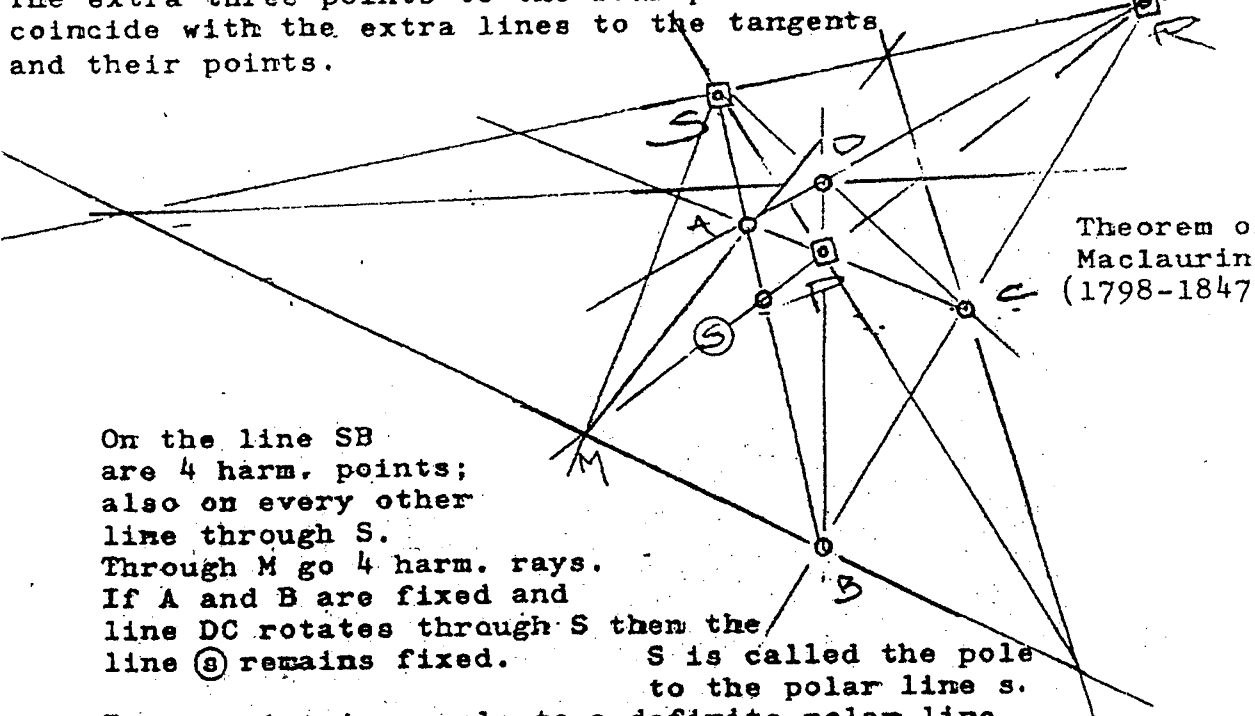
Four tangents on a conic with two touching points produce the Brianchon configuration.





Four points with four tangents on a conic --

The extra three points to the four points and their sides coincide with the extra lines to the tangents and their points.



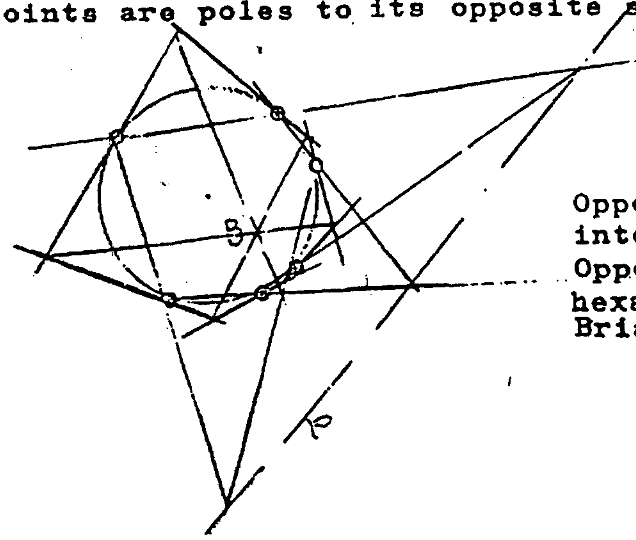
Theorem of Maclaurin (1798-1847)

On the line SB are 4 harm. points; also on every other line through S.
Through M go 4 harm. rays.
If A and B are fixed and line DC rotates through S then the line (s) remains fixed.

S is called the pole to the polar line s.

Every point is a pole to a definite polar line.

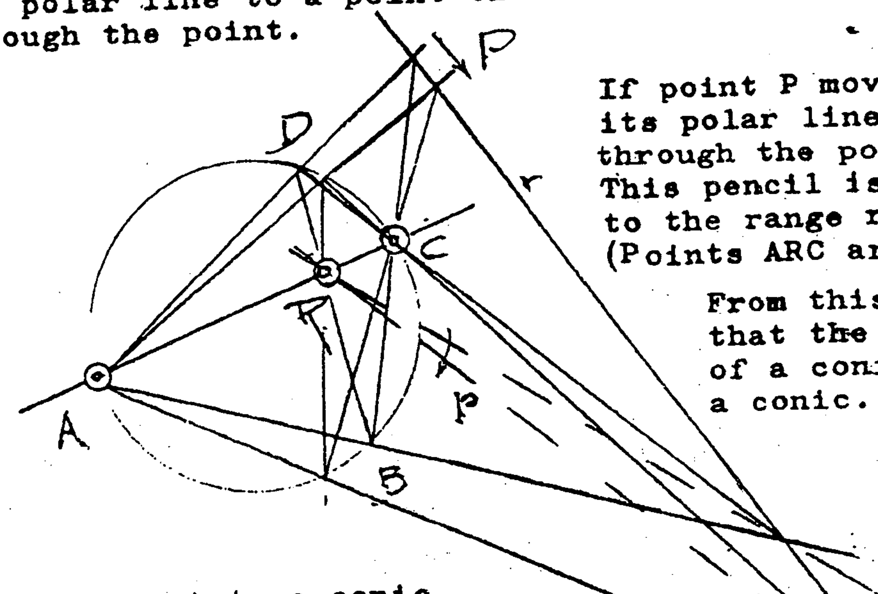
PSR is a self-polar triangle with respect to the conic; its points are poles to its opposite sides.



Hexagon on a circle. The pole to each side is found by tangents in points.
Opposite sides of the hexagon intersect on the Pascal line(p).
Opposite intersections of the hexagram connect through the Brianchon point(B).

The Main Theorem of the theory of Pole and Polar:

- Every point on a line has its polar line passing through the pole of the line.
- Every line through a point has its pole on the polar line of the point.
  (A point is harm. separated from its polar line by the conic. A line is harm. separated from its pole by the tangents from the point to the conic.)
- The polar line to a point on the conic is the tangent through the point.



If point P moves on r its polar line p rotates through the pole R. This pencil is projective to the range r. (Points ARC are fixed)

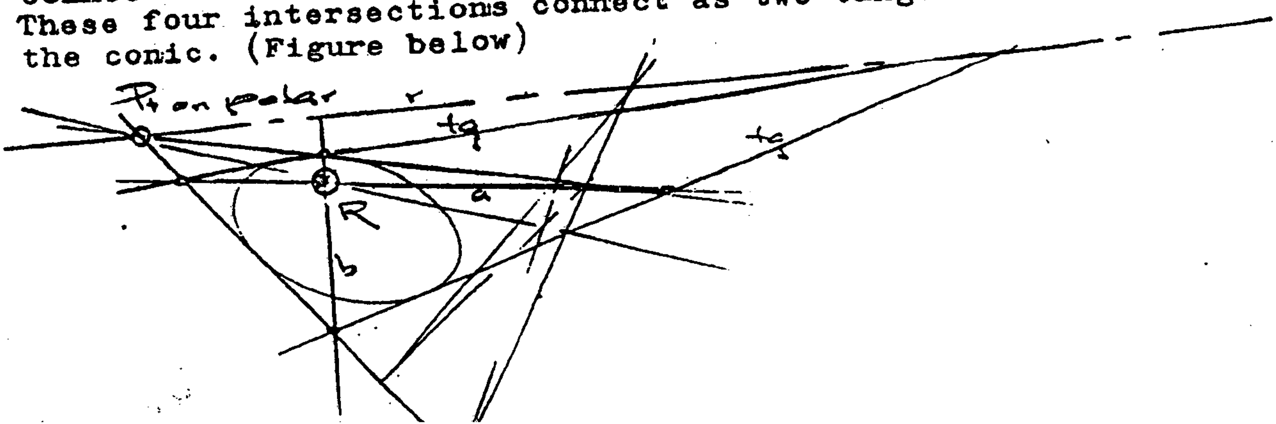
From this we deduce that the polar form of a conic is again a conic.

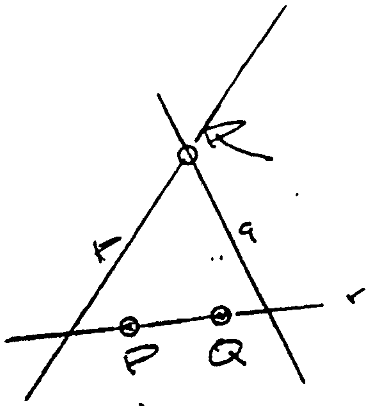
With respect to a conic , two points in the plane of this conic are called c o n j u g a t e points if each point lies on the polar of the other. Two lines are called conjugate lines with respect to the conic if each line goes through the pole of the other.
(A point is conjugate to every point of its polar line.
A line is conjugate to every line through its pole.)
A polar triangle is a triad of conjugate points and lines.
Points on a conic are self-conjugate and tangents are also self-conjugate.

Construction of conic from conjugate points on a line or from conjugate rays through a point (Involution) -

Line through a pole (gives four harm. points): from intersections with conic draw lines to the conjugate points . These intersect on the conic. (Fig. above)

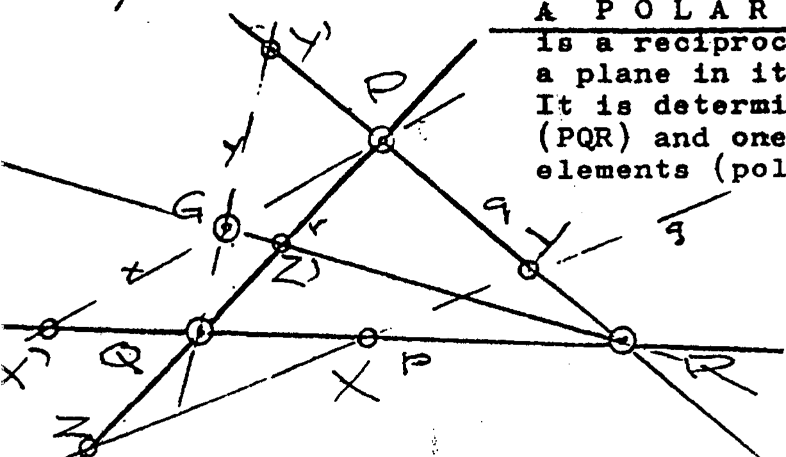
Point on a polar line (gives four harm. lines): two connections with conic intersect with conjugate rays (ab). These four intersections connect as two tangents to the conic. (Figure below)





The connecting line of two poles P and Q is the polar line to the point of intersection of the two polar lines p and q of the points P and Q.

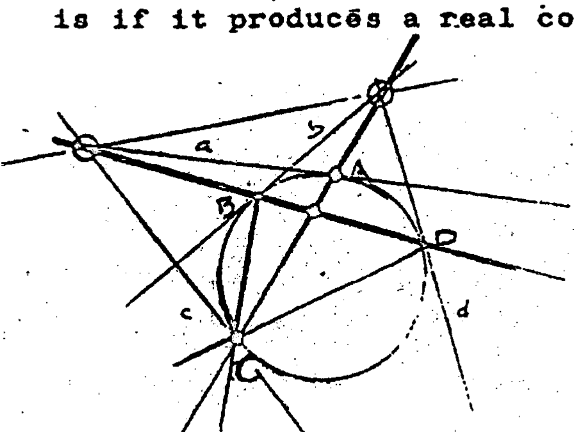
Two conjugate points are separated harm. by the conic (real or imaginary).



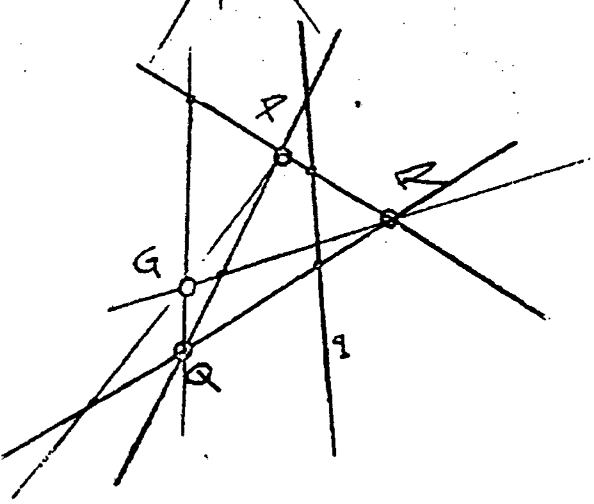
A POLAR SYSTEM is a reciprocal correspondence in a plane in itself. (Point polar to line) It is determined by a polar triangle (PQR) and one pair of corresponding elements (pole G and its polar g).

Xx Yy are corresp. pairs of pole and polar.
Y Y' ,X X' are conjugate pairs.
PQ QR RP are also conjugate pairs.

The three involutions on the sides of the polar triangle are therefore determined. Double elements can be found with Steiner's construction . Connections with the poles give the tangents. (Points on x and y are also to be constructed). With this the conic curve can be drawn -that is if it produces a real conic, and not an imaginary one.



If two conjugate lines intersect the conic, the intersections ABCD are four harm. points and the tangents abcd are four harm. rays.
(On a line through the pole are four harm. points; project them from C onto the conic.
Through a point on a polar are four harm. rays. Intersect them with c,or with any tangent in 4 harm.points.)

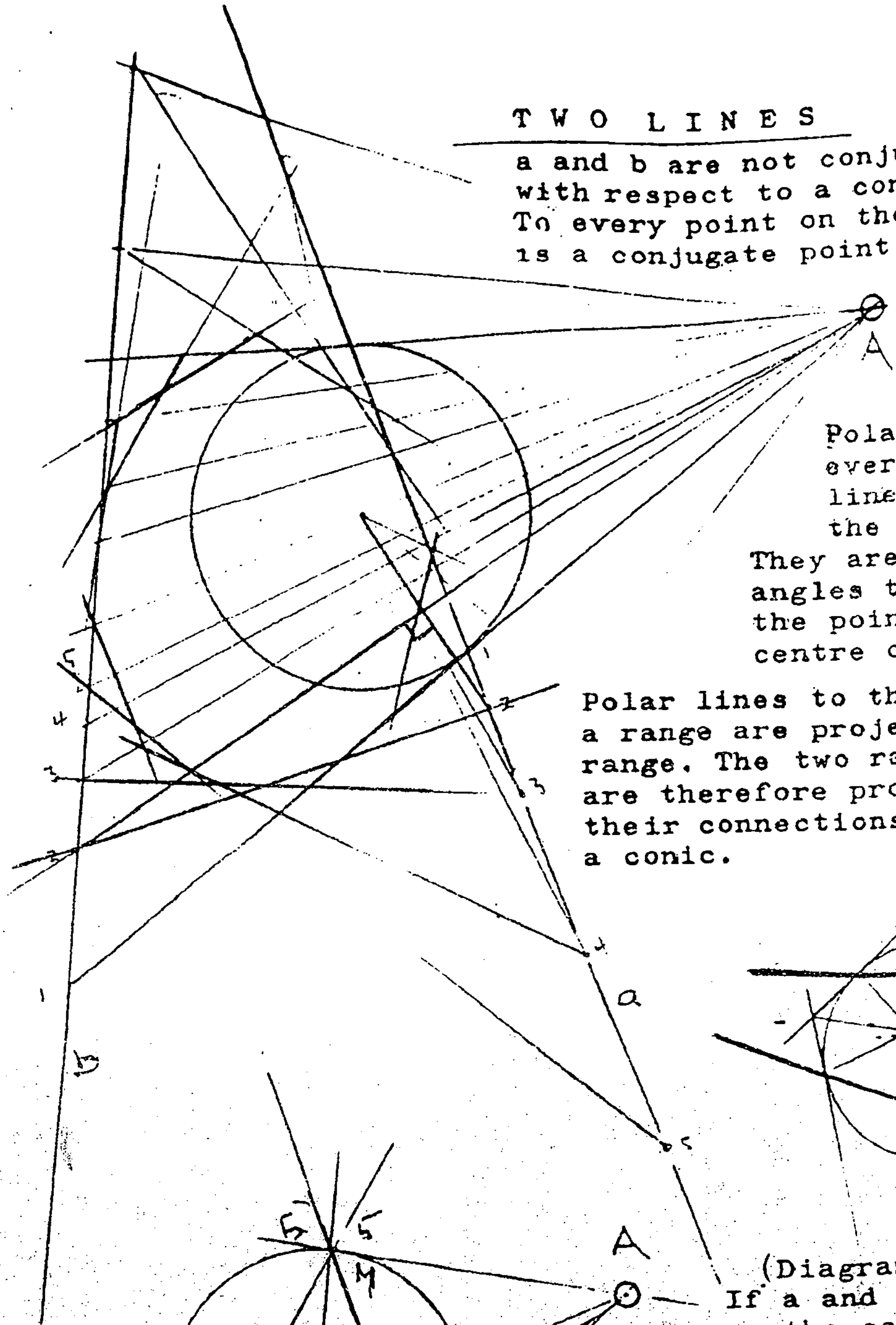


In this figure the 4 pairs of poles and polars are an elliptic polar system and therefore determine an imaginary conic.

The polar g does not go through the section where the pole G is situated.

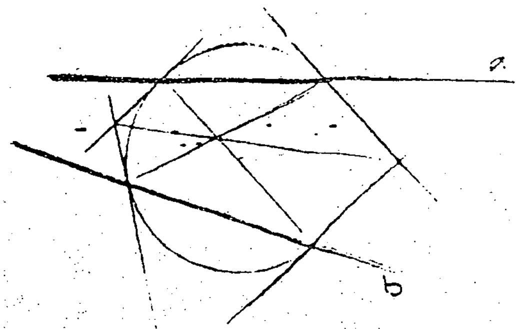
T W O L I N E S

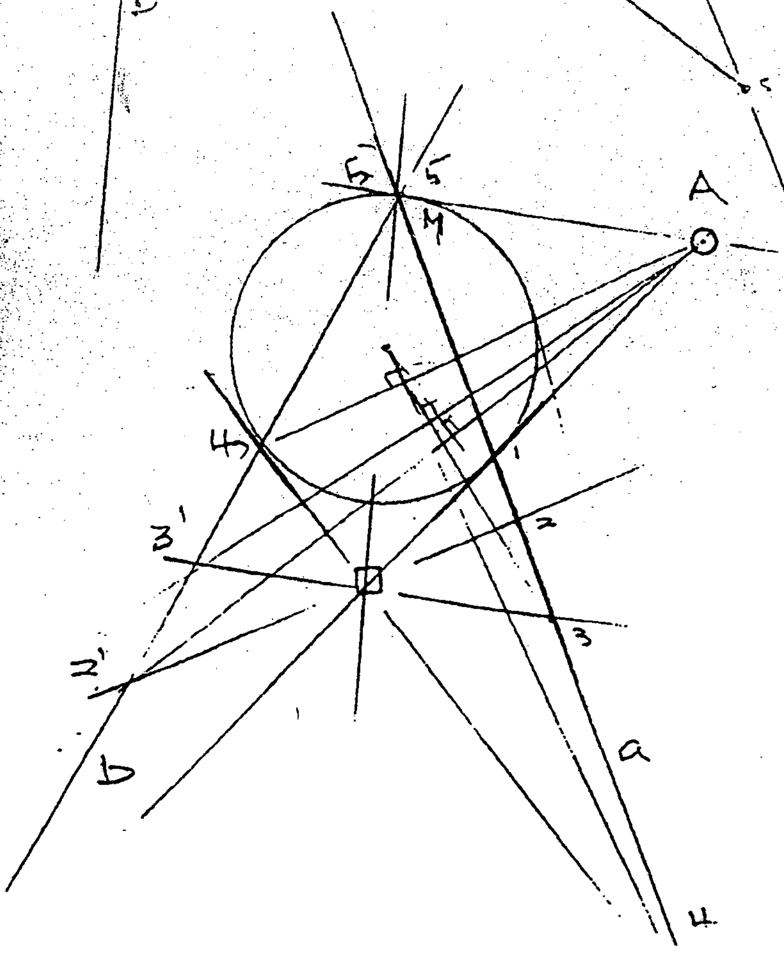
a and b are not conjugate lines with respect to a conic (circle). To every point on the line a is a conjugate point on the line b.



Polar lines to every point on line a go through the pole A of a. They are at right angles to lines from the points to the centre of the circle.

Polar lines to the points of a range are projective to the range. The two ranges a and b are therefore projective and their connections produce a conic.





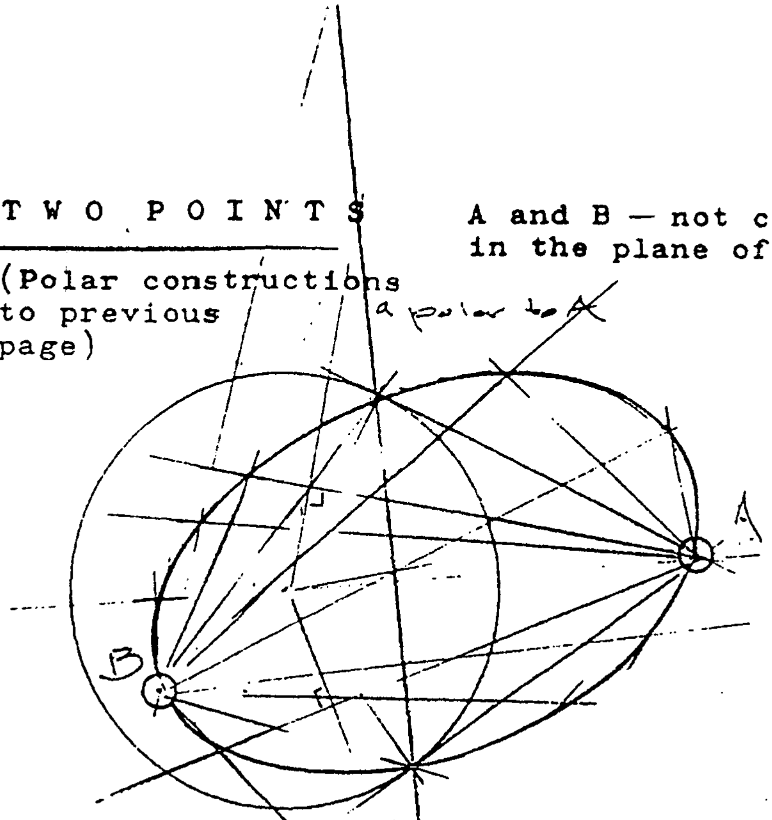
(Diagram on left): If a and b intersect on the conic, the two ranges become perspective. (The point of intersection corresponds to itself) The centre of perspective is the intersection of the two tangents to the second points of a and b on the conic.

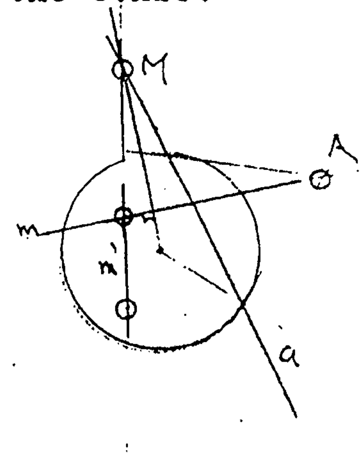
(Diagram above): Two lines intersecting a conic plus the tangents at their intersections, become six tangents to a conic. (Test with Brianchon)

T W O P O I N T S

(Polar constructions to previous page)

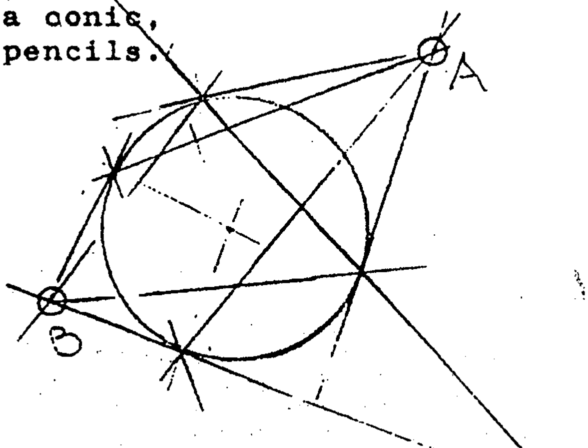
A and B — not conjugate points in the plane of the conic.





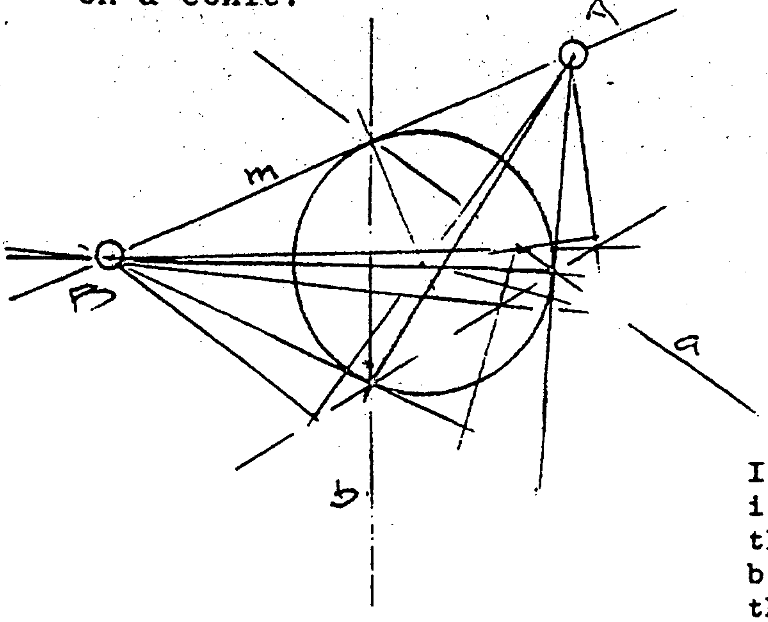
To every line through A is a conjugate line through B. Their intersections are on a conic, as they are two projective pencils.

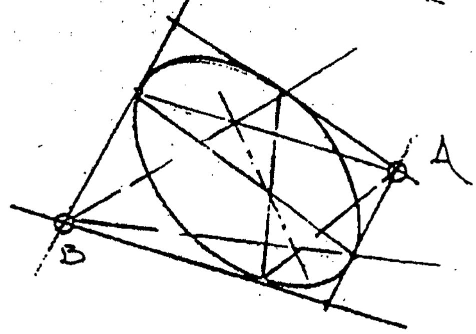
(Poles on a are projective to the pencil in A. The pencil in B is perspective to the range a.)



Intersections of two pairs of tangents to a conic (A and B) and the four touching points are all six points on a conic.

Test with Pascal line :



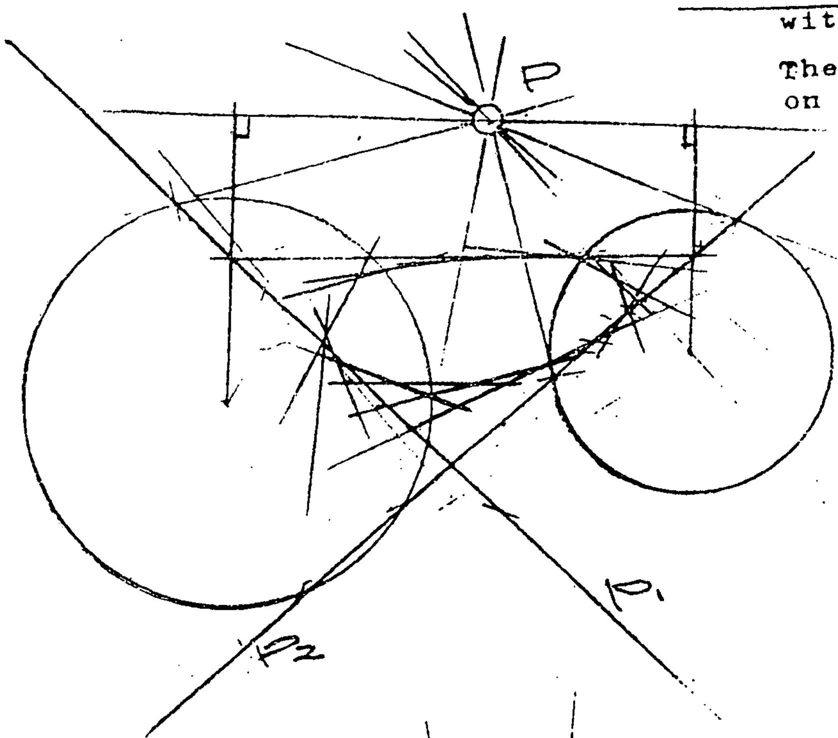


If connection of A with B is a tangent to the conic, this becomes m common to both pencils ; they are then perspective.
The intersection of the two pencils is a straight line.

POLES to a pencil P with respect to two circles.

They are projective ranges on the two polar lines p_1 and p_2, (the two polar lines of point P).

They produce a conic.

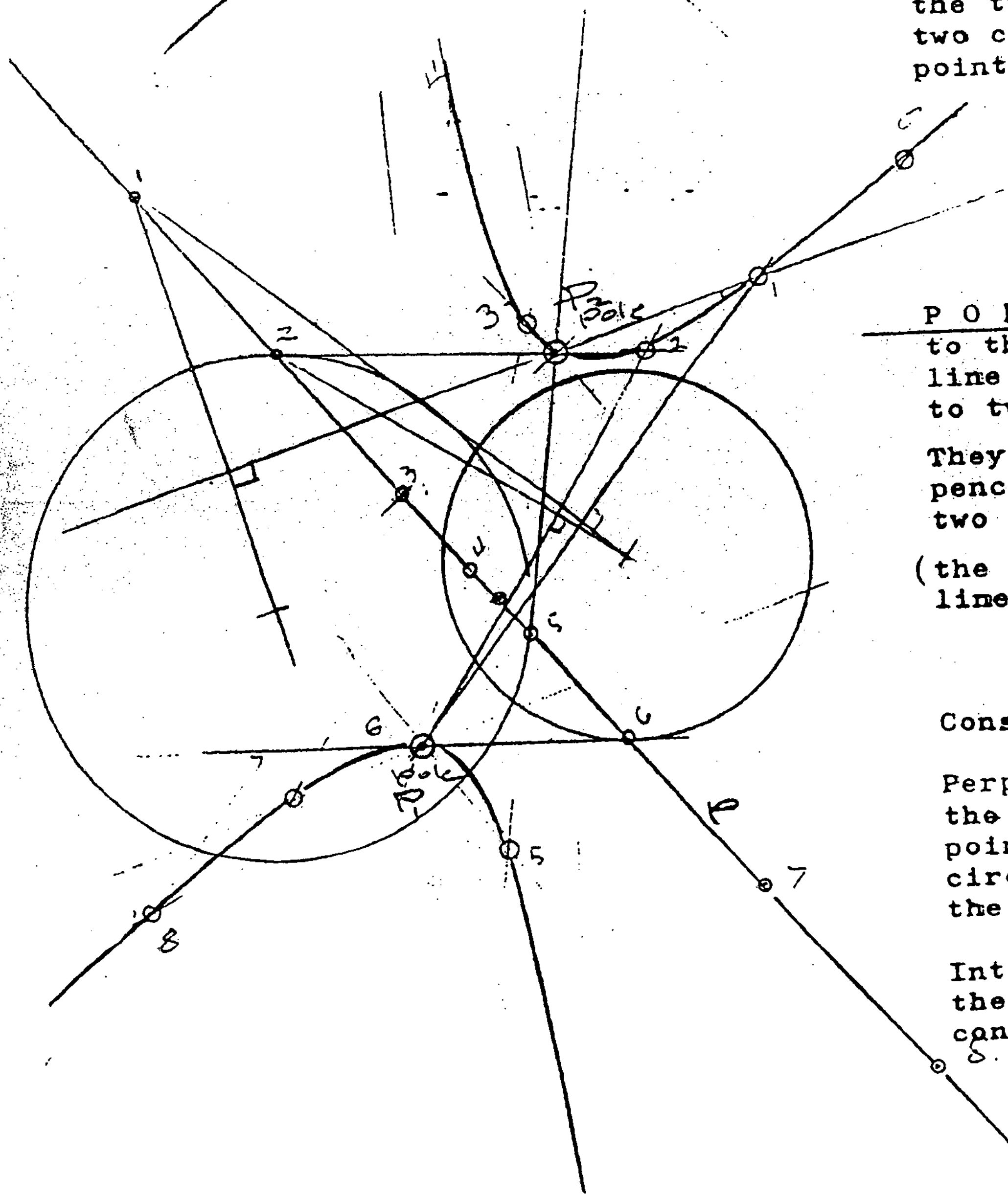


Construction:

Perpendiculars (to the ray through P through the two centres of the circles) intersect the two polars in two corresponding points.

POLAR LINES to the points of a line p with respect to two circles.

They are projective pencils through the two poles P_1 and P_2, (the two poles of the line p).



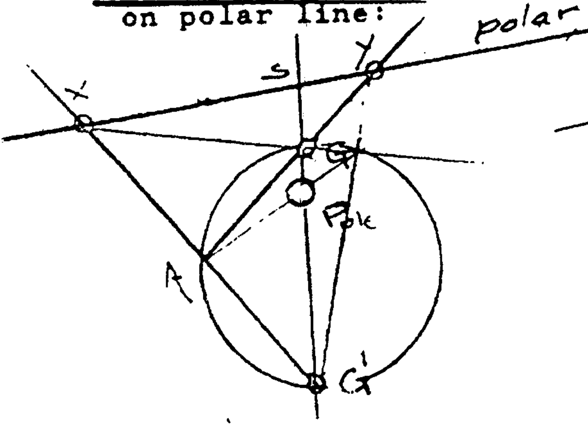
Construction:

Perpendiculars (to the lines connecting points of line p with circles' centres) throug the poles P_1 and P_2.

Intersections are the points of the conic.

From: Projective Geometry by B.C.Patterson (New York 1937)
Chapter: Theory of pole and polar in Projective Geometry

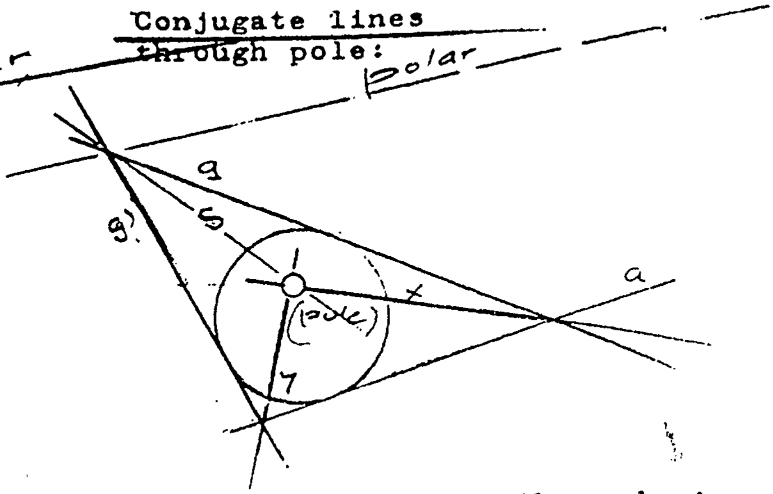
Conjugate points on polar line:



On a line through the pole: 4 harmonious points GG'PS

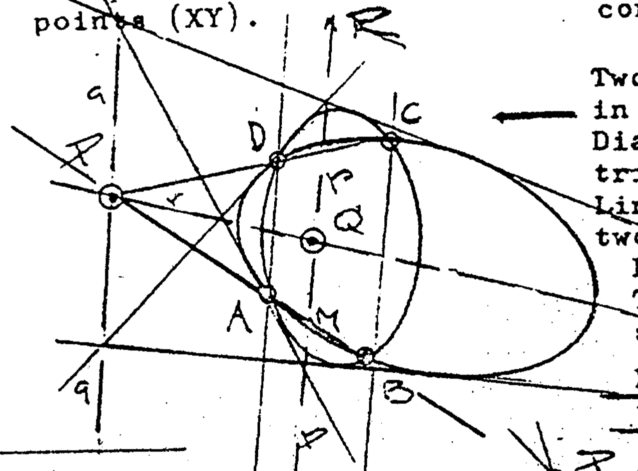
The connections from any point on the conic (A) to G and G' intersect the polar in two conjugate points (XY).

Conjugate lines through pole:



Through a point on the polar: 4 harmonious lines gg'ps

The intersections of any tangent to the conic (a) with g and g' connect with the pole in two conjugate rays(xy).



Two conics with four tangents in common:

Diagonals form self-polar trilateral (pqr) .
Line PAM must intersect the two conics in their intersection B , for PAMB are harm.4 points. The same applies to the other sides of the quadrängle ABCD.

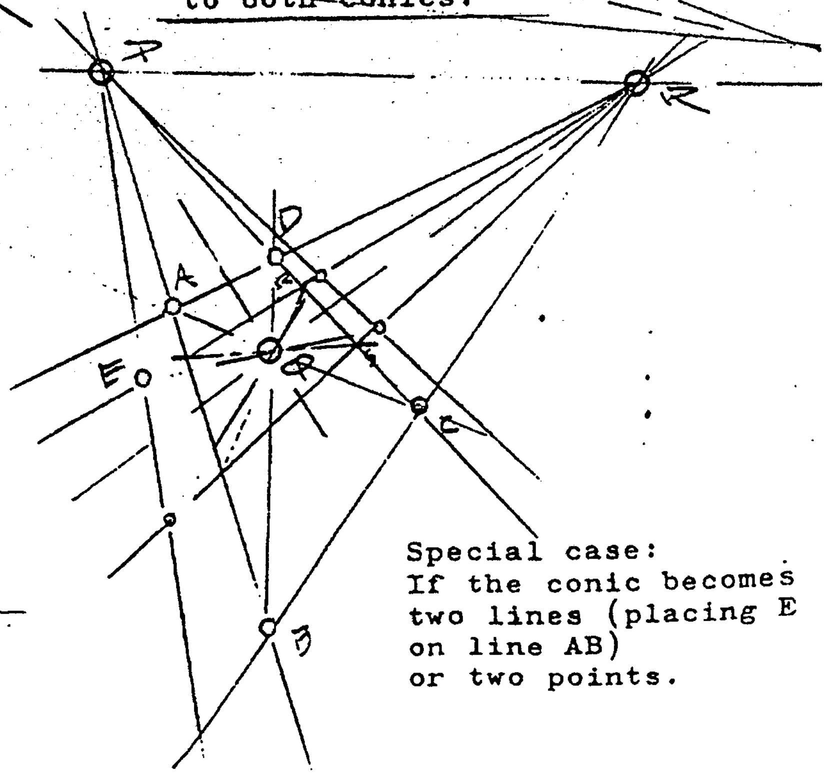
P Q R is the self-polar triangle to both conics.

Two quadrangles with the same diagonal triangle ---- have their eight points on one conic.

Five points determine a conic . On the sides of the quadrangle are always four harmonious points.

Polar :

Two quadrilaterals with the same diagonal trilateral --- have their eight sides as tangents to the same conic.



Special case:
If the conic becomes two lines (placing E on line AB) or two points.

QUADRATIC TRANSFORMATION I

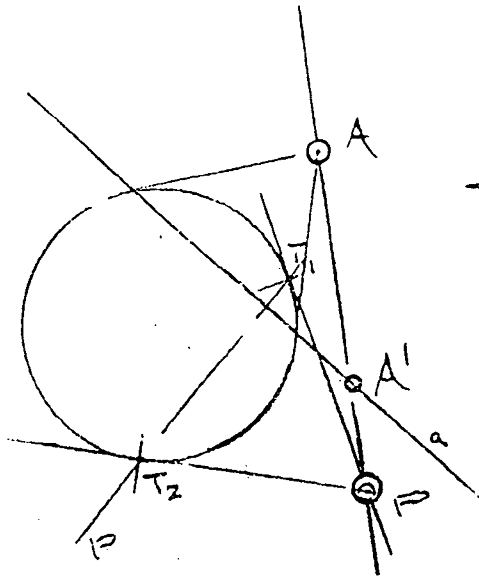
Given : a fixed conic and a fixed point P.

To point A corresponds point A'. A' is the intersection of line PA and polar line a of A.

A and A' are conjugate points and are collinear with the fixed point P. The correspondance is involutary (interchangeable).

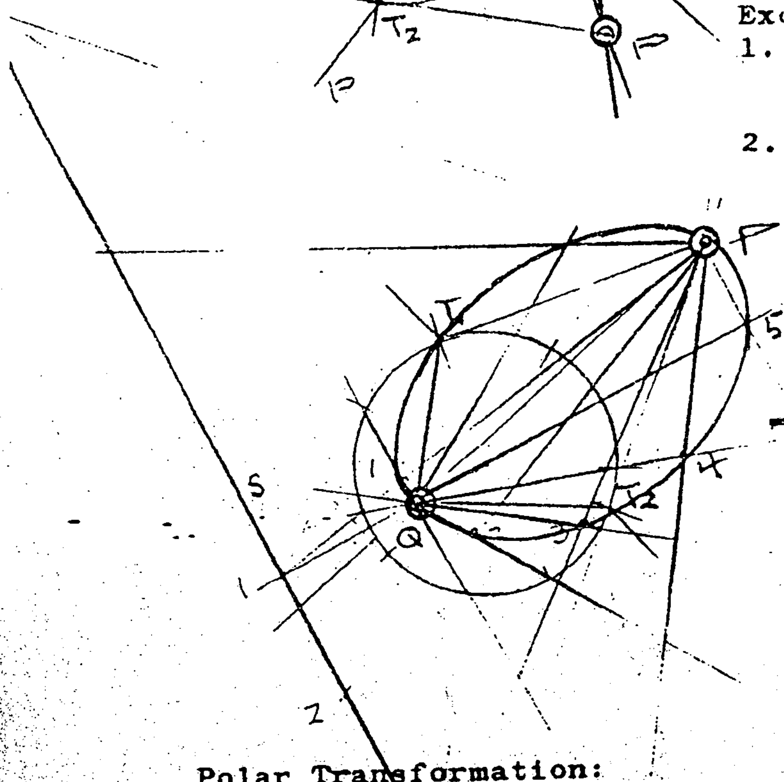
Exceptional points (above diagram):

1. A any point on polar to P. A' coincides with P. If A=P: A' is indeterminate.
2. T_1 and T_2 (contacts of tangents from P) : corresp. points are indeterminate. Any point on PT_1 has its corresp point as T_1 (on PT_2 -- as T_2).



Points on a line have their corresponding points on a conic.

They are the intersections of two projective pencils; the pencil in P and the pencil in the pole of the line s are projective to the range of the line.



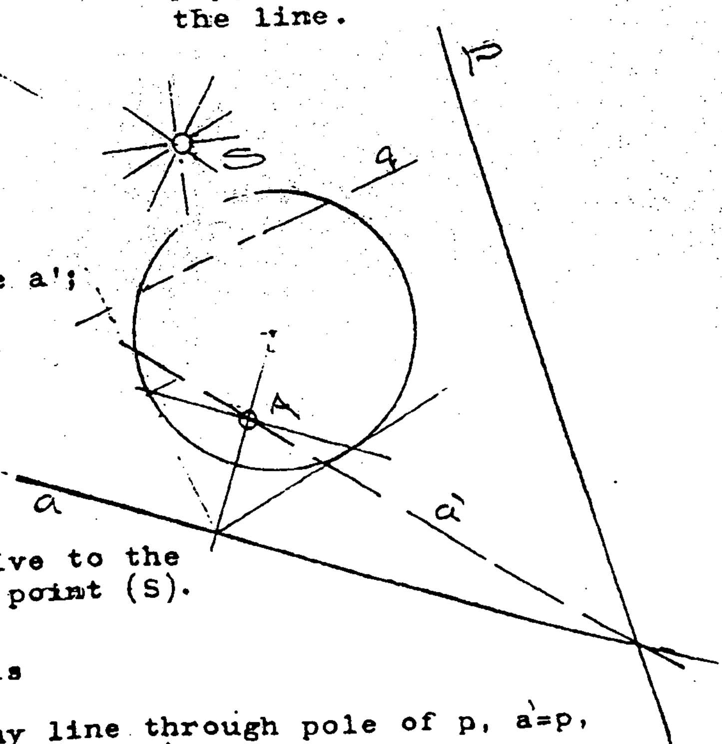
Polar Transformation:

given: a fixed conic and a fixed line p.

To line a corresponds line a'; a and a' intersect on p; a' goes through pole of a.

Lines through a point have their corresponding lines as an envelope to a conic.
The range on p is projective to the range on the polar of the point (S).

a and a' are conjugate rays. The correspondence is involutary.
Exceptional lines: a any line through pole of p, a'=p, (etc. as above)



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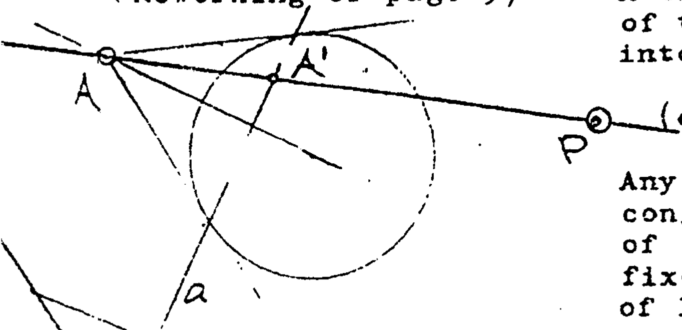
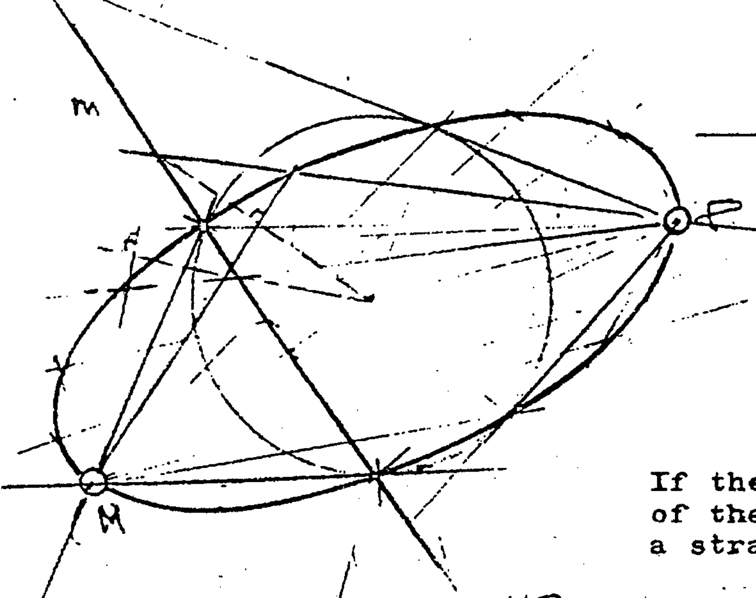
QUADRATIC TRANSFORMATION

(Reworking of page 9)



A one-dimensional form of the first order is transformed into a one-dimensional form of the second order (e.g. range into conic).

Any point A becomes its conjugate point A' on the ray of the pencil through the fixed point P. (Intersection of line A-P with polar line)

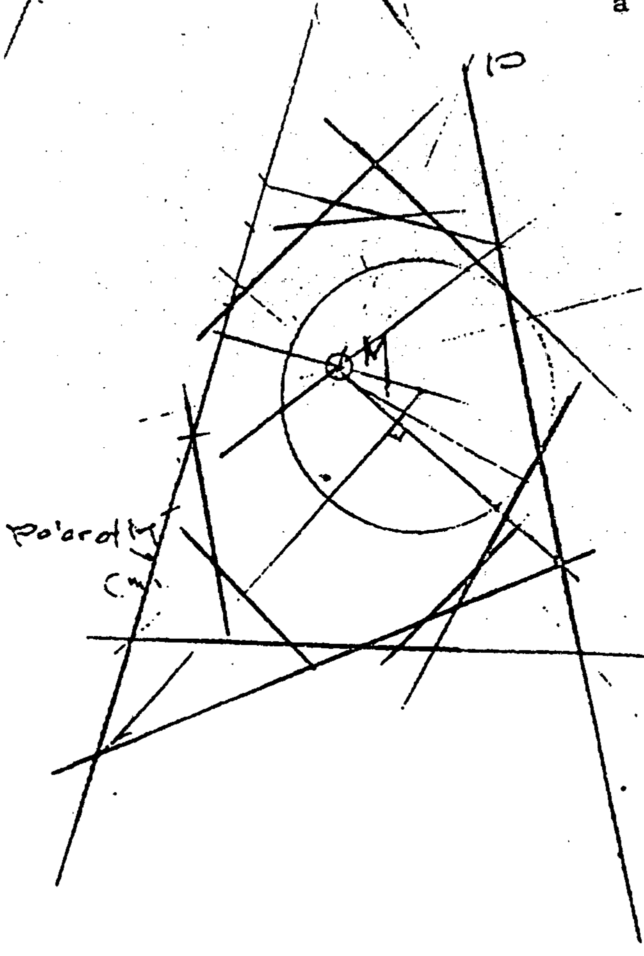


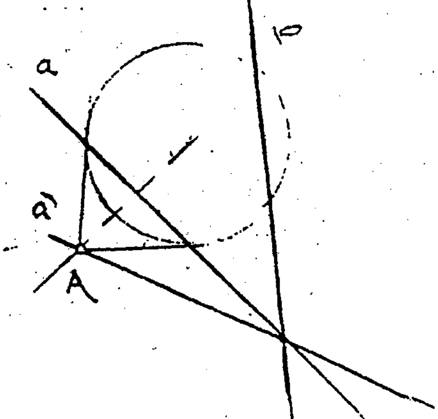
Transformation of a range m of the first order.

On every ray of P construct the conjugate point to the intersection with line m.

The resulting conic passes through P and M (pole to m) and through the touching points of the tangents to the circle from P and M.

If the line M to P is a tangent of the circle, the conic becomes a straight line.

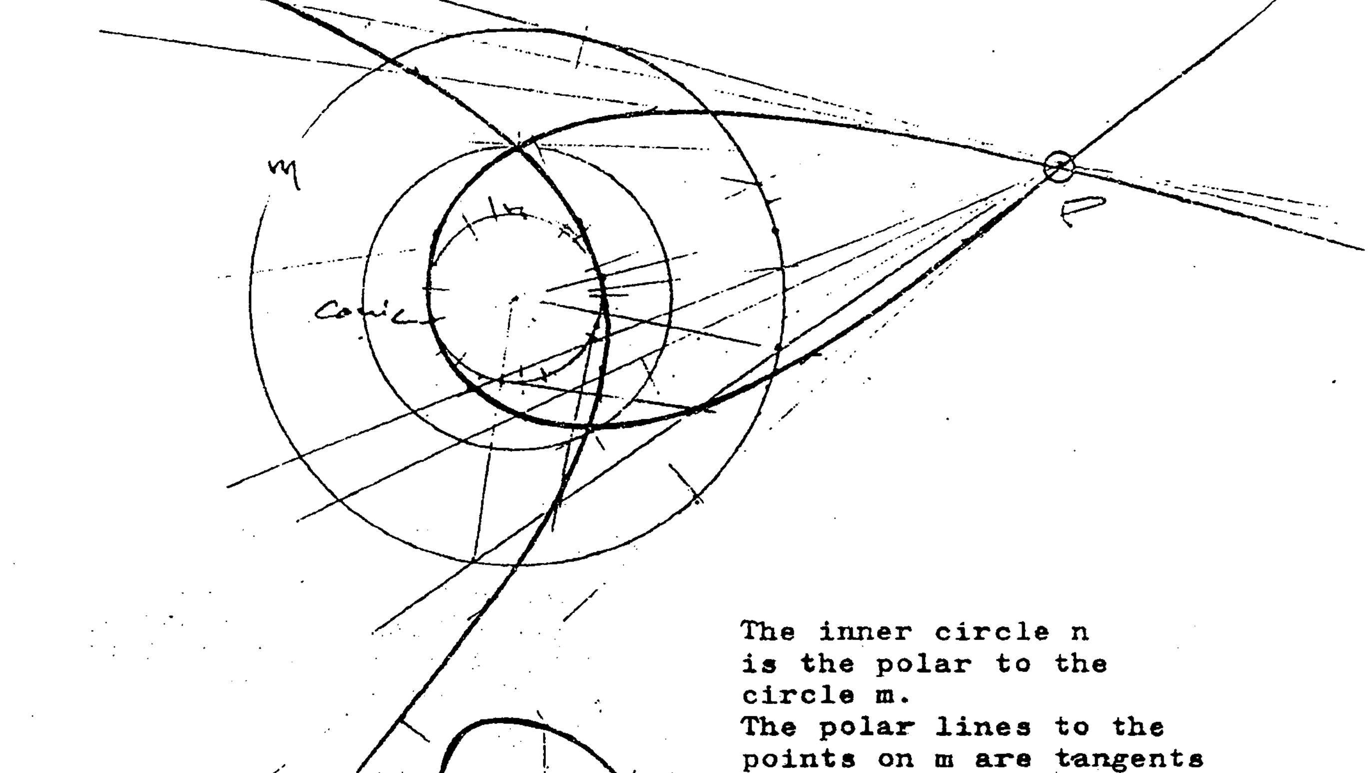




Polar to the above:

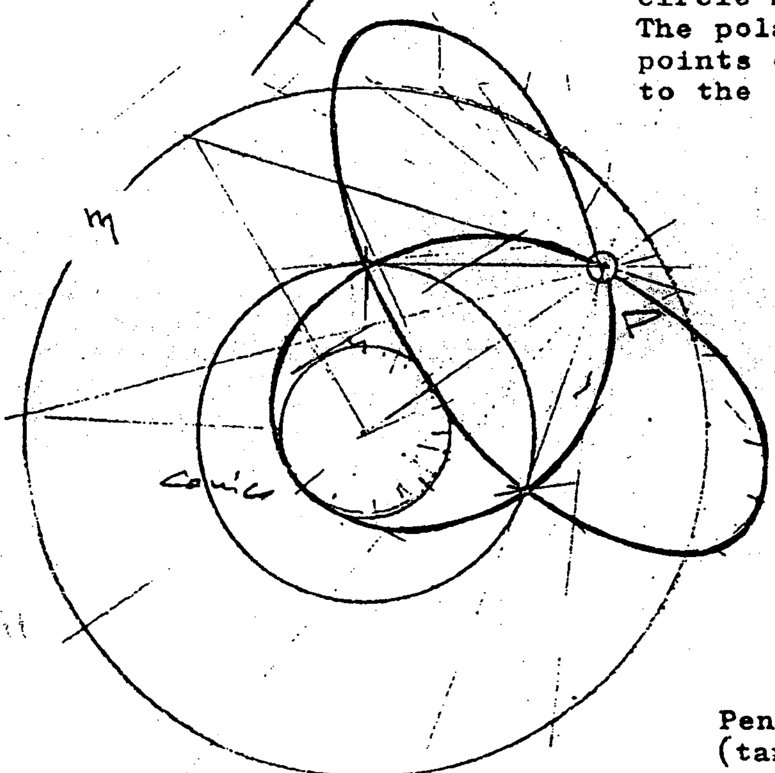
A line p and a point M are given. The transformation is from a pencil of the first order to a pencil of the second order. Through every point on p the conjugate line to the line through M is constructed.

QUADRATIC TRANSFORMATION
of a circle m
with respect to a conic
and a pencil through P.



The inner circle n
is the polar to the
circle m.
The polar lines to the
points on m are tangents
to the circle n.

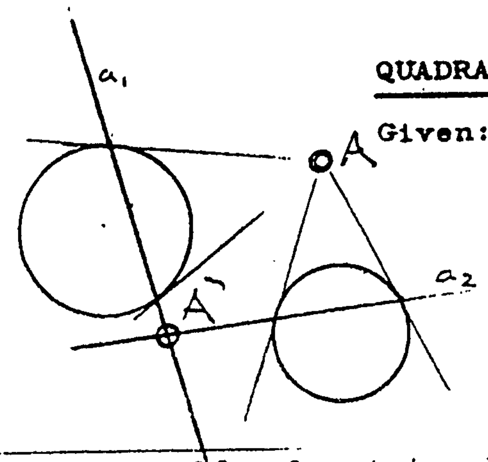
The resulting curve
lies on the inter-
sections of the polar
lines with the
corresponding lines
through P.



Pencil of second order
(tangents to circle n)
intersects with pencil
through P.

Suggestion: Find the curve if
point P is at infinity.

QUADRATIC TRANSFORMATION II



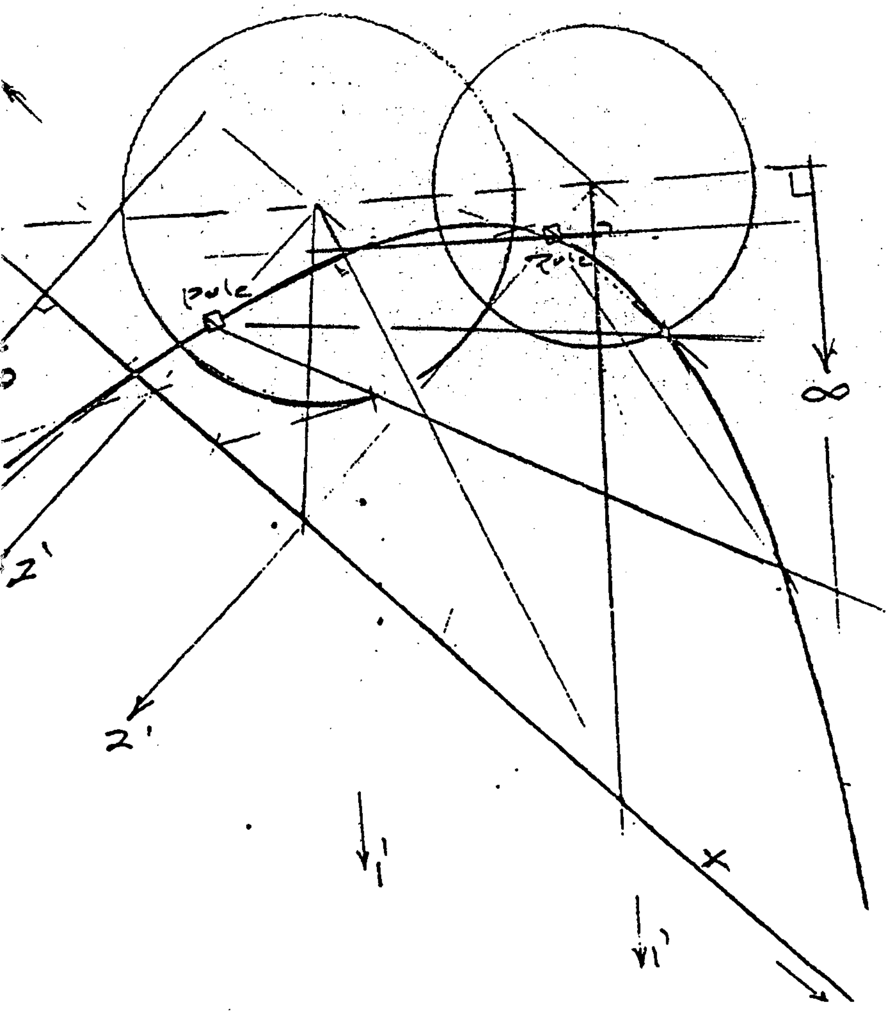
Given: two fixed conics.

To point A corresponds A'. A' is the intersection of the two polar lines with respect to the two conics. A and A' are conjugate points with respect to each of the two conics.

If a common self-polar triangle exists : (PQR) and
if A = P, then A' is any point on p (QR).
If A is any point on p then A' coincides with P.
Similar for q (PR) and r (PQ).

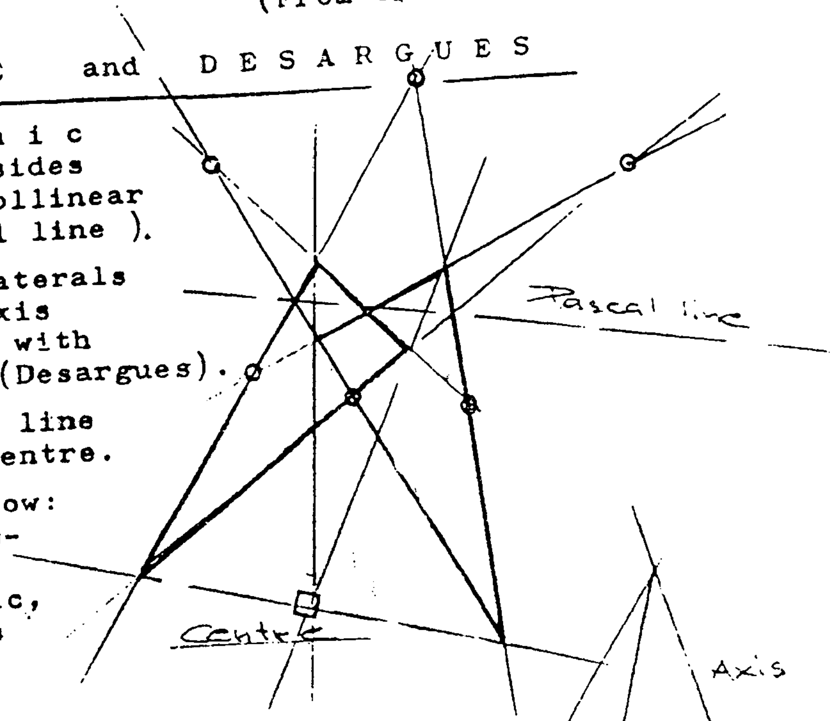
Points on a line have their corresponding points on a conic. This conic goes through the vertices of the common self-corresponding triangle of the two conics.

Polar lines form two projective pencils through the two poles of the line x with respect to the two given conics.



(From Cremona, Art.155)

PASCAL LINE and DESARGUES



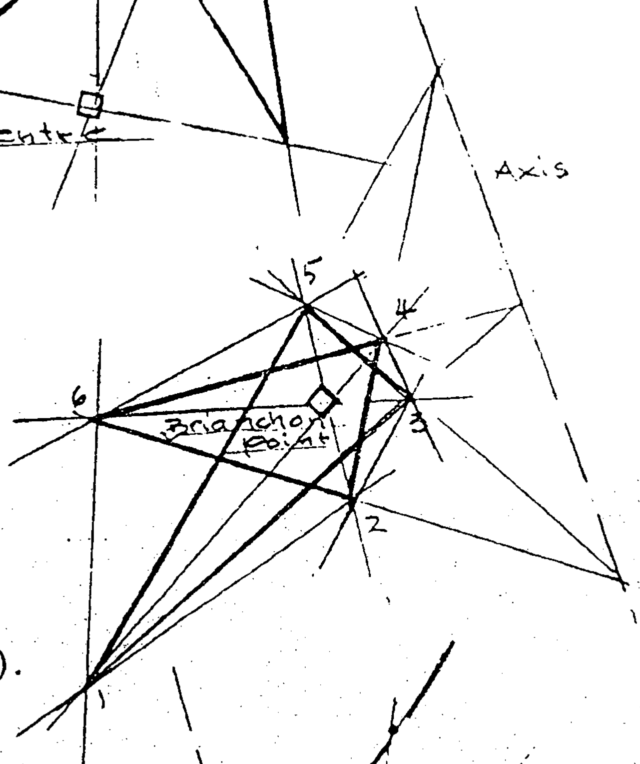
Six points on a conic (hexagon); opposite sides intersect in three collinear points (axis = Pascal line).

Two perspective trilaterals with respect to an axis are also perspective with respect to a centre (Desargues).

If there is a Pascal line there must exist a centre.

The two triangles show: intersections of non-corresponding sides are points on a conic, if the two triangles are perspective.

BRIANCHON POINT and DESARGUES

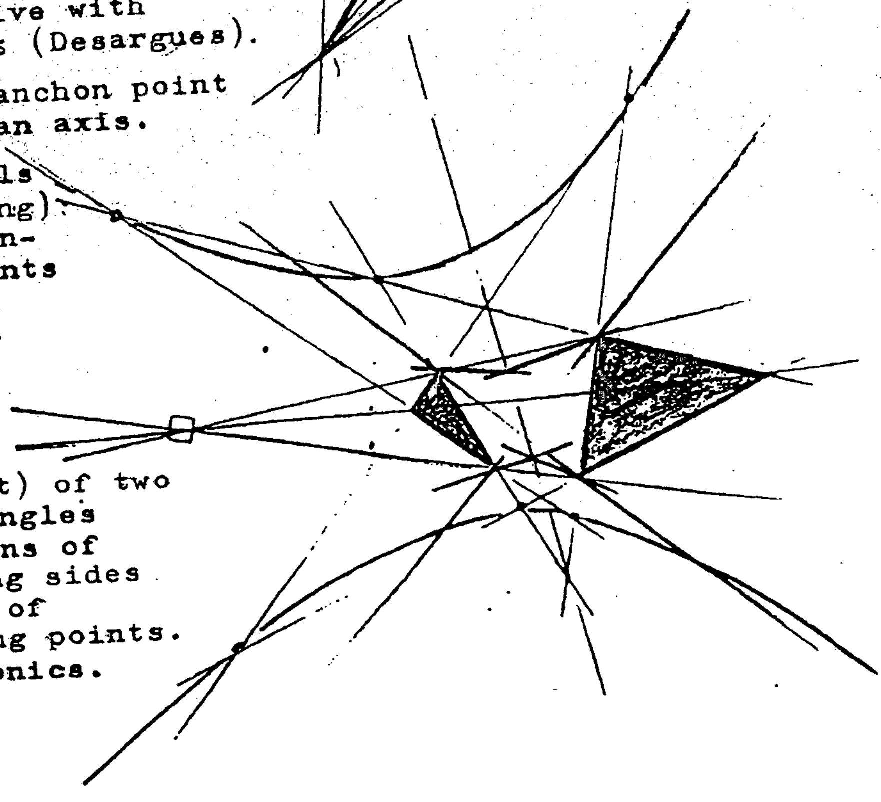


Six tangents to a conic (hexagram); opposite intersections connect with three collinear limes (centre=Brianchon point).

Two perspective triangles with respect to a centre are also perspective with respect to an axis (Desargues).

If there is a Brianchon point there must exist an axis.

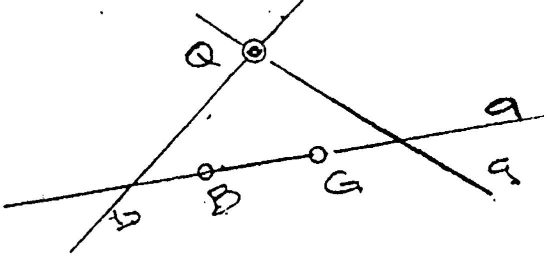
The two trilaterals show(middle drawing) connections of non-corresponding points are tangemts to a conic, if the two trilaterals are perspective.



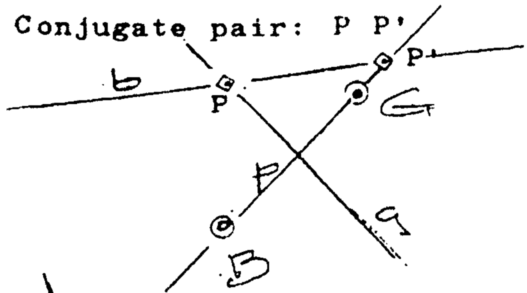
Example (at right) of two perspective triangles with intersections of non-corresponding sides and connections of non-corresponding points. They form two conics.

(From Cremona, Art.326)

Poles and polars: gb - GB



Conjugate pair: P P'

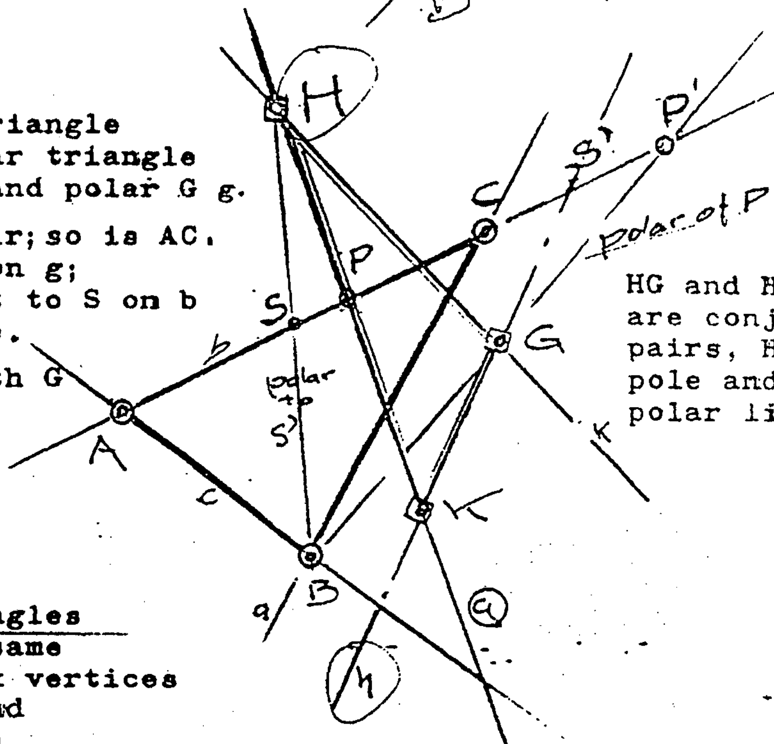


Construction of
second self-polar triangle
from given self-polar triangle
A B C abc and pole and polar G g.

PP'is a conjugate pair; so is AC.
Choose any point H on g;
find conjugate point to S on b
with auxiliary circle.

Connection of S' with G
is polar line to H.

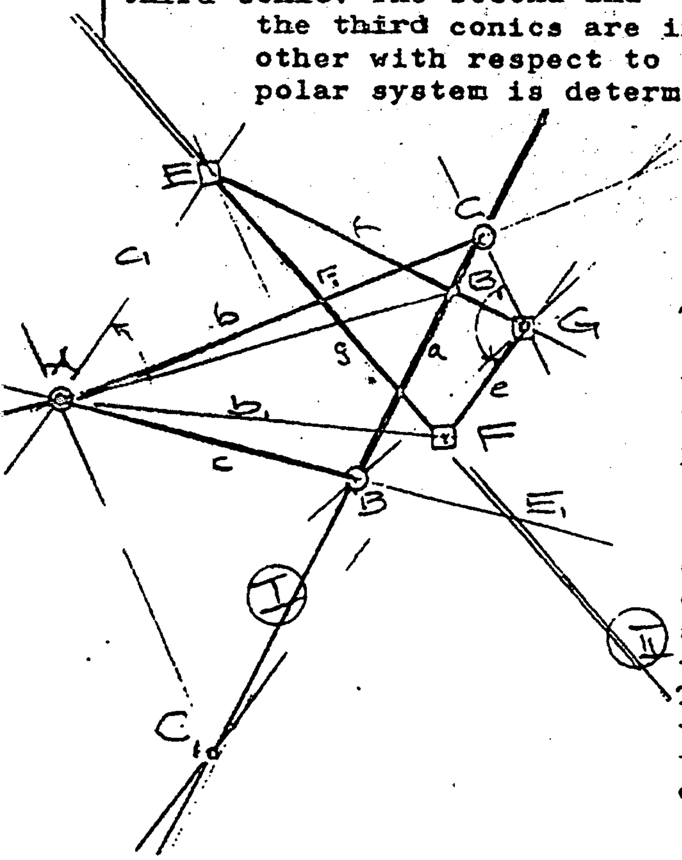
H G K is a second
self-polar triangle.



HG and HS'
are conjuga
pairs, H h
pole and
polar line.

THEOREM :

Two self-polar triangles
with regard to the same
conic have their six vertices
on a second conic and
their six sides on a
third conic. The second and
the third conics are in polar opposition to each
other with respect to the first conic, whose
polar system is determined by the self-polar triangle
ABC and the pole and polar
G and g.



Pole and polars:
Aa Bb Cc Ee Ff Gg.
B_1b_1 C_1c_1

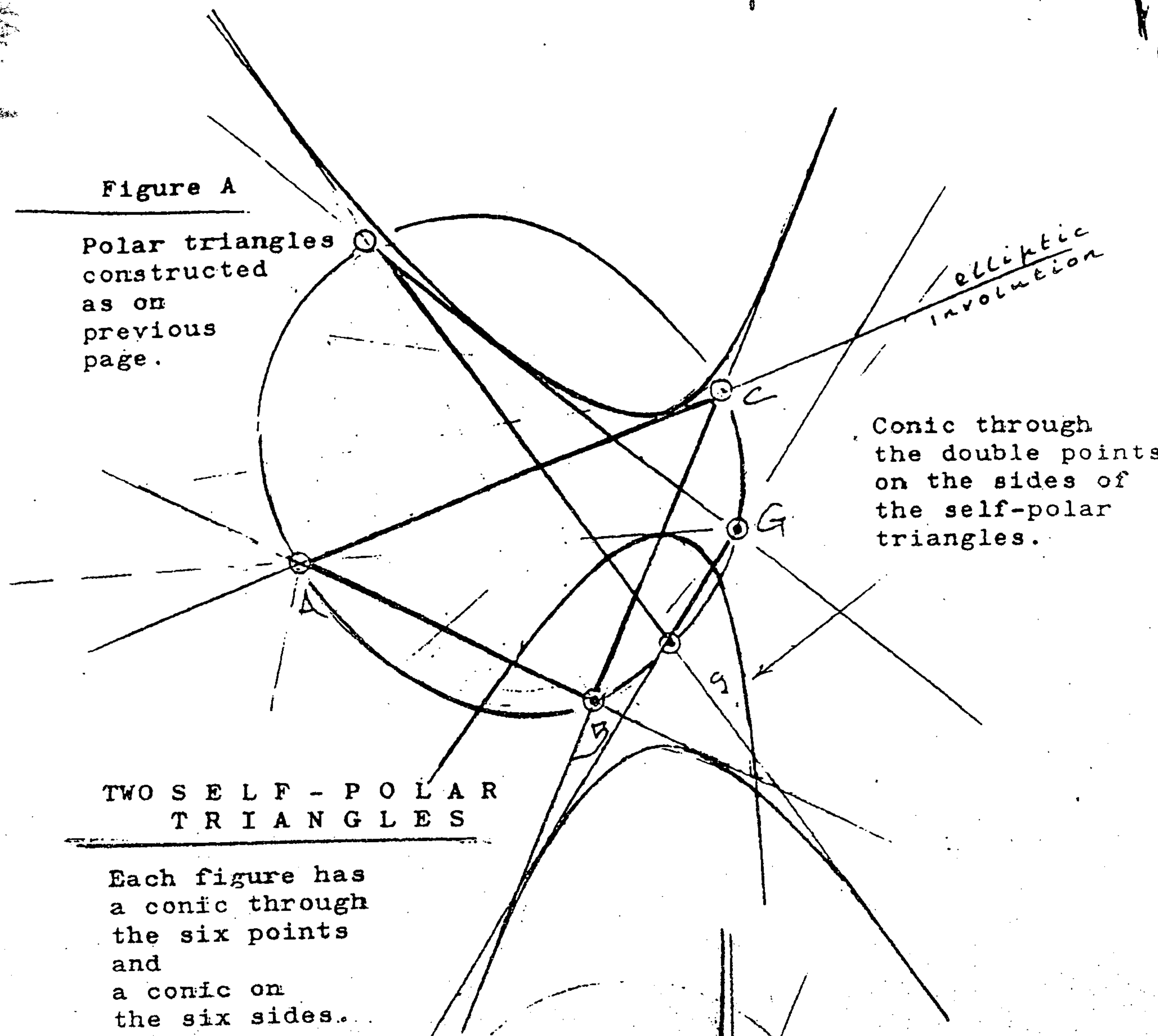
The pencil in G is projective
to the pencil in A: cb, bc_1 are
the polar lines to
the range CB, BC_1

Projective ranges:
I: CB, BC_1 II: $E_1FF_1E = F_1EE_1F$
(pairs change over)

$CF_1 - B_1E - BE_1 - C_1F$ are tangents.
CB and EF are tangents, as well
as projective ranges, and connect
to form the above tangents.
The six vertices are the poles
to the sides of their own
triangles, therefore they must be
on a conic as well.

Figure A

Polar triangles constructed as on previous page.



Conic through the double points on the sides of the self-polar triangles.

TWO SELF-POLAR TRIANGLES

Each figure has a conic through the six points and a conic on the six sides.

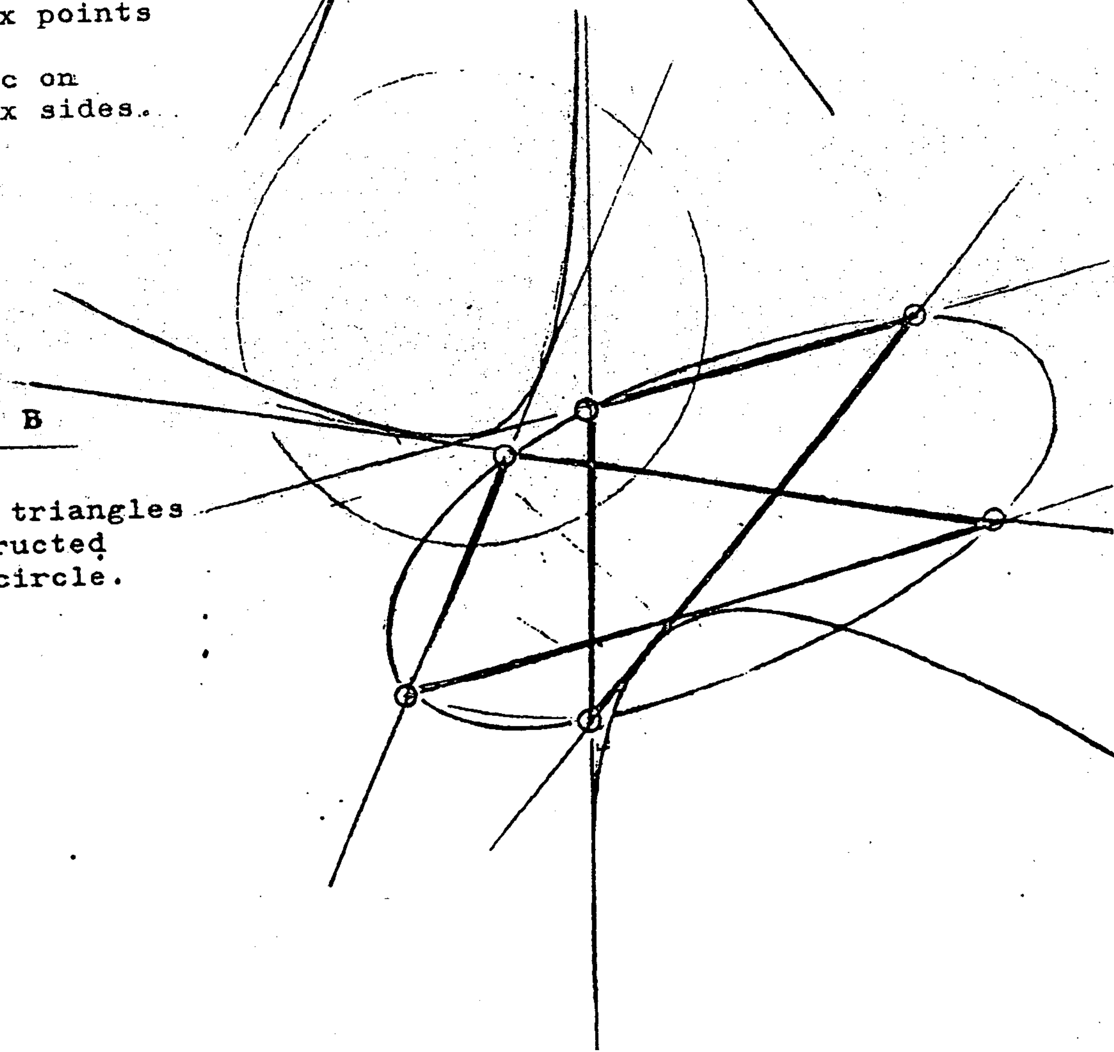
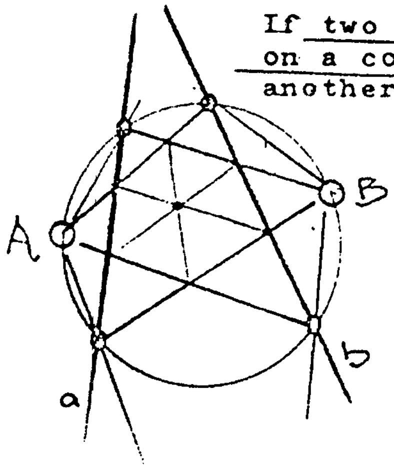
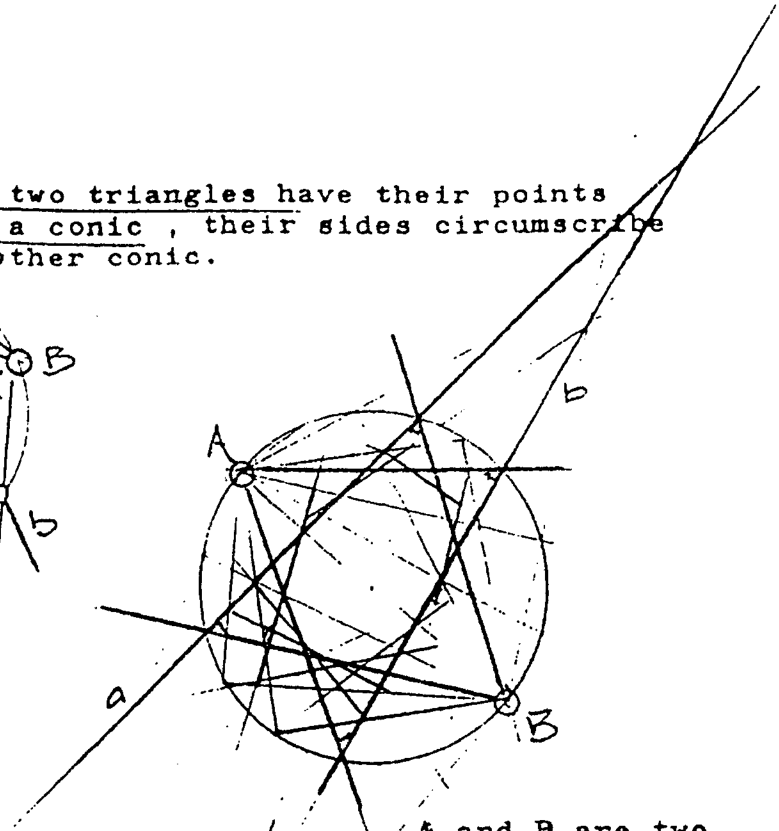


Figure B

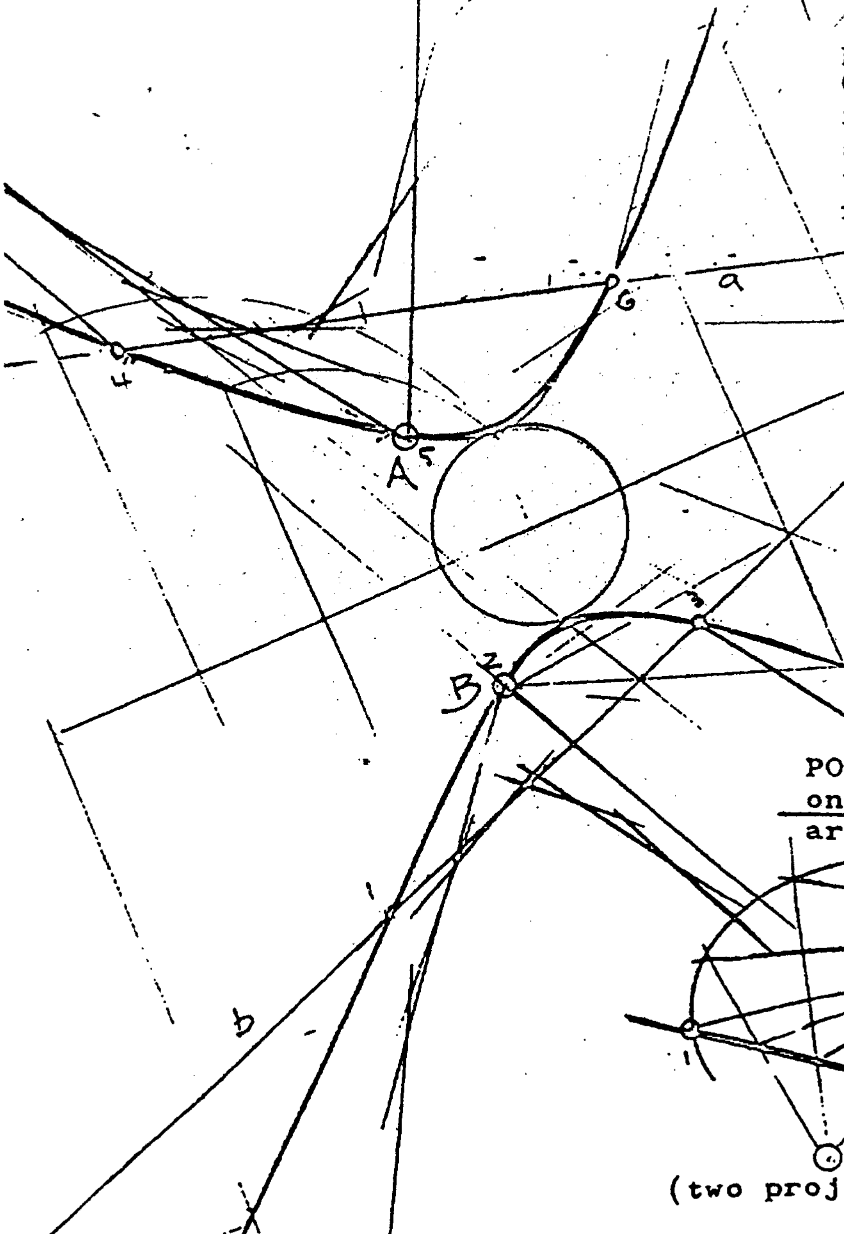
Polar triangles constructed to a circle.



If two triangles have their points on a conic, their sides circumscribe another conic.



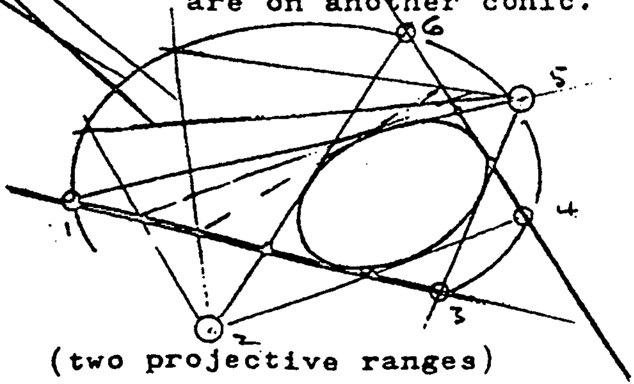
A and B are two projective pencils. Corresponding rays intersect the two sides a and b in two projective ranges.



Six points on a hyperbola.

The six sides are tangents to a conic.

POLAR: Two trilaterals on a conic - the 6 points are on another conic.



(two projective ranges)

From: Projective Geometry by L.N.G.Filon (London 1908)

SYSTEMS of CONICS

All conics passing through four fixed points ABCD are called a p e n c i l of conics.

A pencil of conics determines an involution on any straight line in the plane of this pencil.

Through a point P of the line goes a definite conic of the pencil (as 5 points determine a conic) which crosses the line in a second point P'.
P' is determined by P and conversely P is determined by P'.

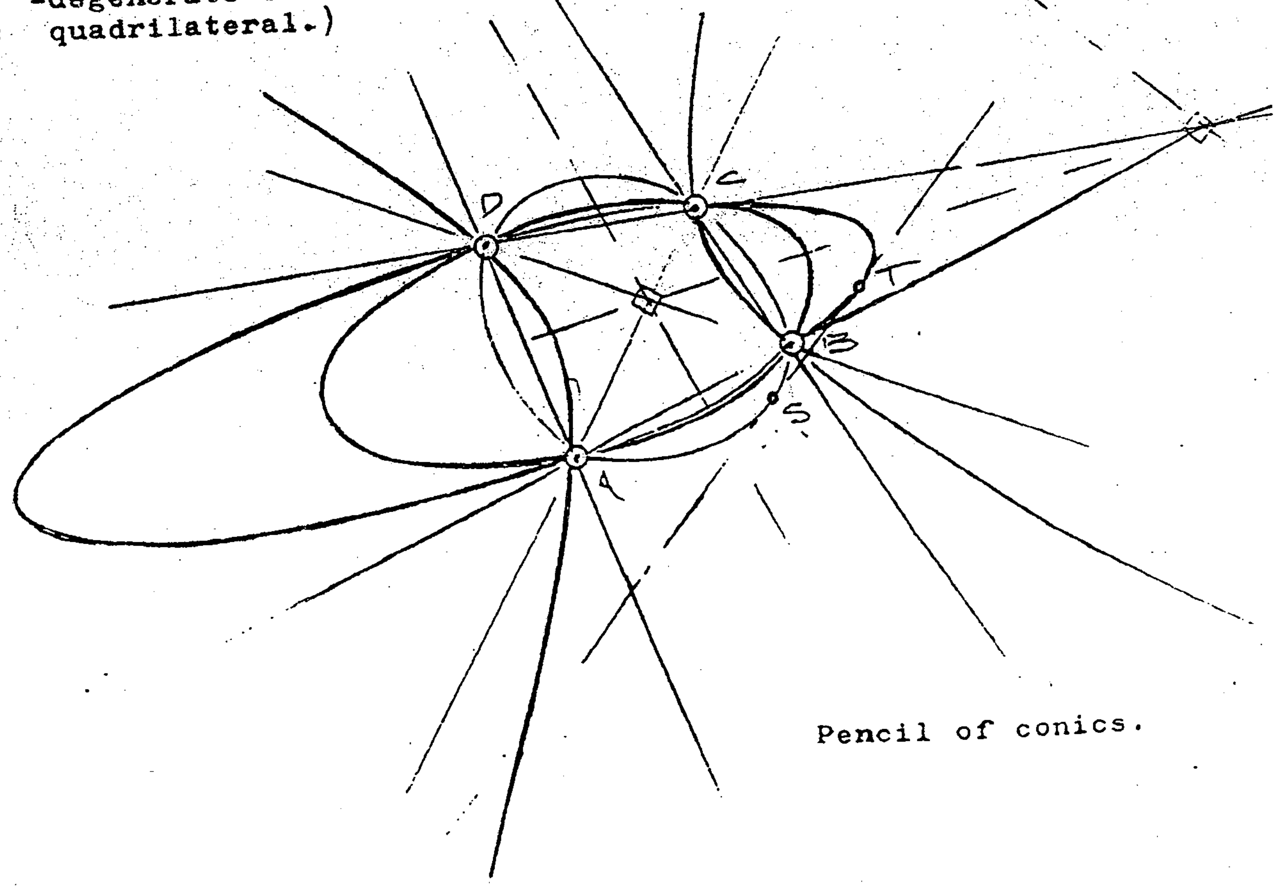
This is a one-to-one correspondence between two ranges on the line; moreover an interchangeable correspondence which makes it into an involution.

The double points are the points of contact of two conics touching the line in T and S.

There are three degenerate conics in the pencil, each one consisting of two opposite sides of the quadrangle ABCD.

Thus the theorem:
Six sides of a quadrangle intersect a line in their plane in three pairs of corresponding points of an involution;
and the polar opposite:
Six points of a quadrilateral connect to a point with three pairs of corresponding lines of an involution.

A r a n g e of conics consists of all the conics touching four fixed lines abcd.
(Polarizing the above considerations, find the:
-involution of tangents to the range from a point.
-degenerate conics through three point-pairs of the quadrilateral.)



Pencil of conics.

COMMON SELF-POLAR TRIANGLE
of a pencil of conics or of a range of conics

The diagonal triangle of the quadrangle ABCD or the diagonal trilateral of the quadrilateral abcd is a self-polar triangle to all the conics of the pencil or to all the conics of the range.

CONJUGATE POINTS with regard to a pencil of conics.

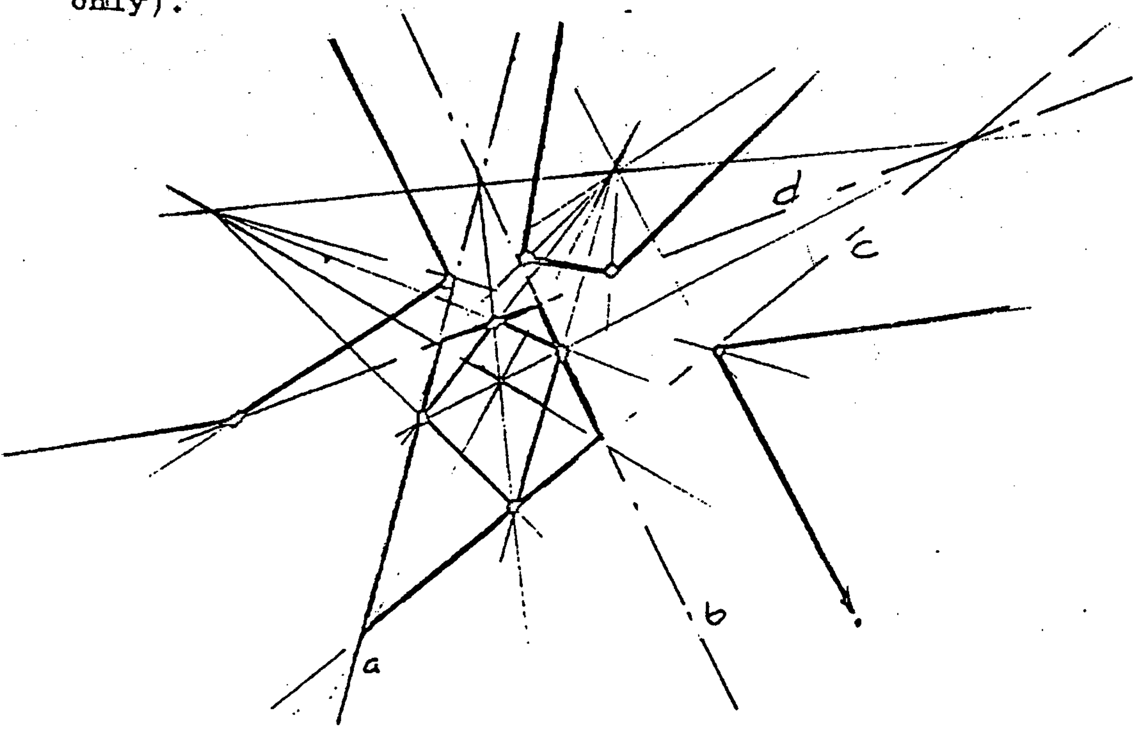
The double points S and T on a line produced by a pencil of conics are harmoniously conjugate with regard to P P' in which the line is cut by any conic.

To every point S is a conjugate point S' in the plane of the pencil of conics.
Consider the conic (of the pencil) through S, and the tangent s in S. A second conic of the pencil touches the line s in the point S'. On any other line through S, point S would not be a double point.

The three points of the self-polar triangle are exceptional as all the points on the opposite lines are conjugate points.
The vertices of this triangle are the only points with the property of having an infinite number of conjugate points. Hence this extra triangle, which is the only self-polar triangle to every conic of the pencil.

Consider the polar opposite: in the field of a range of conics every line has a conjugate line with regard to the range of conics.

Four lines abcd. Range of conics consists of all the conics touching these four lines. In each four-sided section are conics. In this figure are indicated the three harmoniously inscribed conics (touching points only).



The ELEVEN-POINT CONIC

Conjugate points to all the points on a straight line q with regard to a pencil of conics form a conic curve.

This conic goes through eleven points:

- the 3 points of the self-polar triangle X Y Z
- the 6 points on the sides of the quadrangle (on each side is the harmonious conjugate between the corners with respect to the intersection with the line)
- the 2 double points on the line q - T'T.

v_1 and v_2 : Any two conics of the pencil.

Q_1 Q_2 : The two poles of the line q with regard to v_1 v_2.

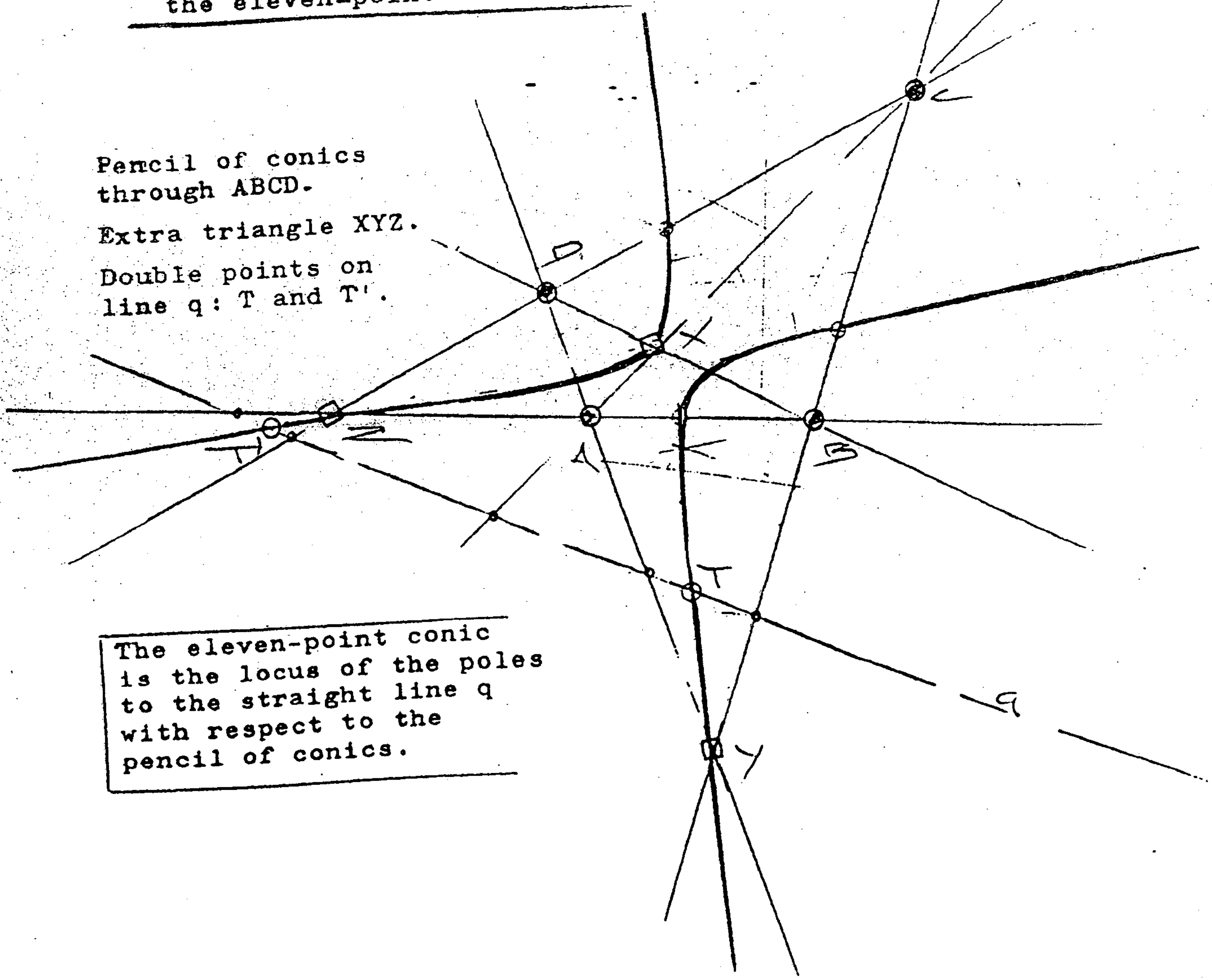
s_1 s_2 : The polars of a point S on q with respect to the two conics.

All points on a polar line are conjugate to the pole - points of s_1 and points of s_2 are conjugate to S.

s_1 s_2 : intersect in point S'. (S' is conjugate with respect to both conics).

S moves on q - s_1 and s_2 rotate to produce projective pencils.

S' therefore traces out a conic passing through Q_1, Q_2, the eleven-point conic.



Pencil of conics through ABCD.

Extra triangle XYZ.

Double points on line q : T and T'.

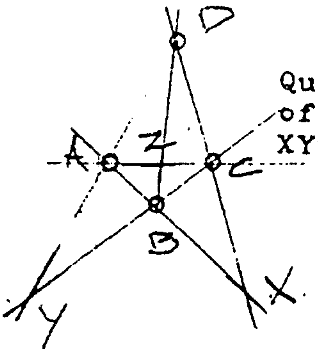
The eleven-point conic is the locus of the poles to the straight line q with respect to the pencil of conics.

Construction of COMMON SELF-POLAR TRIANGLE to two conics

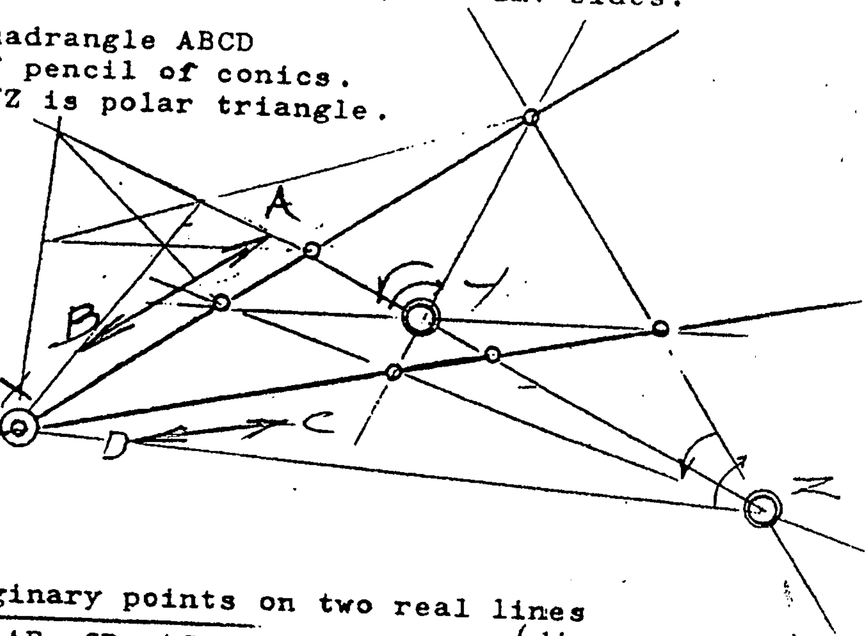
ABCD : points of intersection of the two conics.

If ABCD all real or all imaginary
- the common polar triangle is real.

If AB real , CD imaginary
- one real point, one real side, two im. points, two im. sides.



Quadrangle ABCD
of pencil of conics.
XYZ is polar triangle.

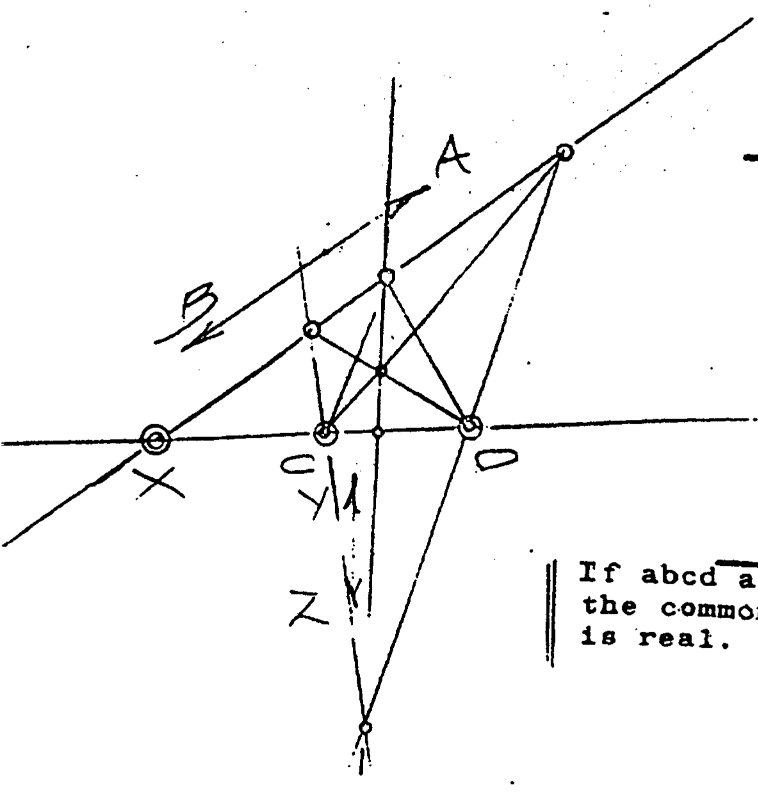


A B C D : imaginary points on two real lines

Of the six lines AB CD AC BD AD BC : (diagram above).

AB and DC are real, AC and BD through Z - imaginary
AD and CB through Y - imaginary.

Extra points of the quadrangle:
AB-DC : X AC-BD : Z CB-AD : Y



A B : imaginary points
C D : real points

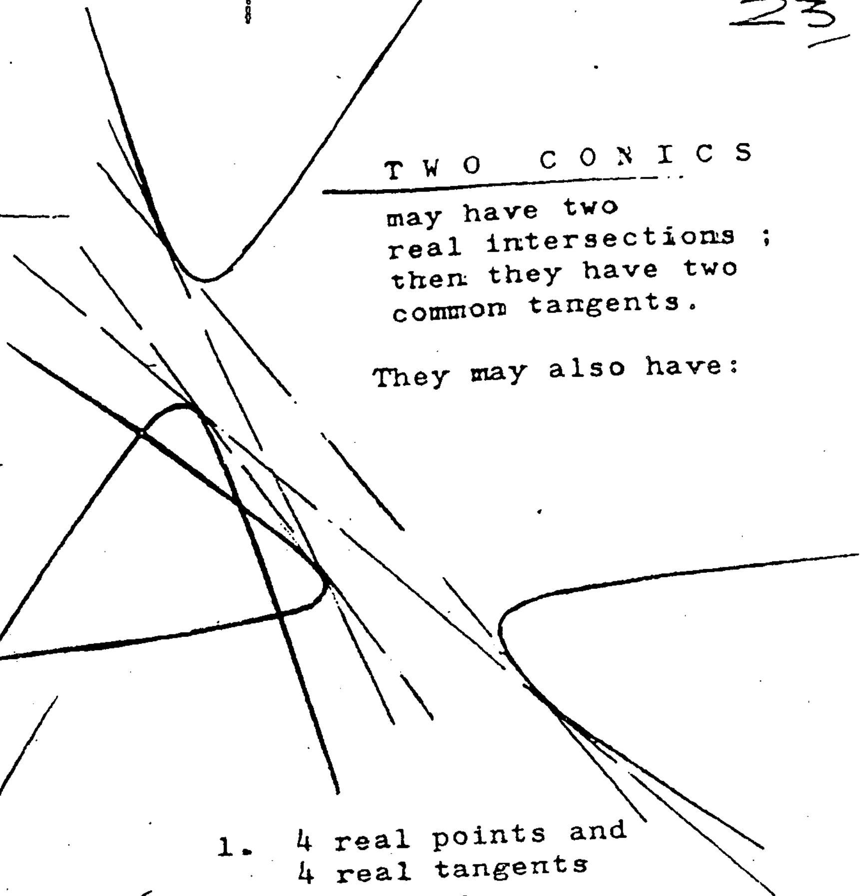
X : AB-DC real

Y : AD-CB imaginary

Z : AC-BD imaginary

Consider in similar ways the self-polar trilateral from four tangents to a range of conics.

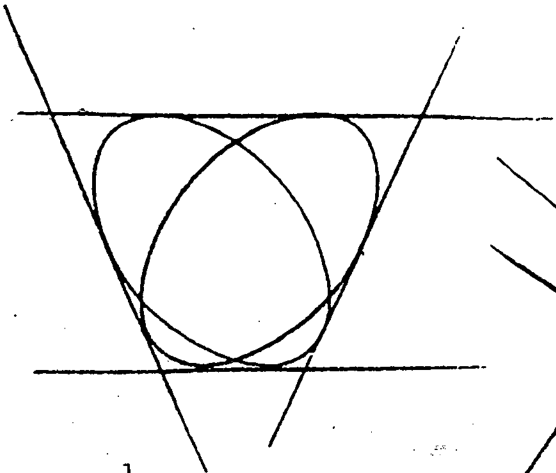
If abcd all real or all imaginary, the common self-polar trilateral is real.



TWO CONICS

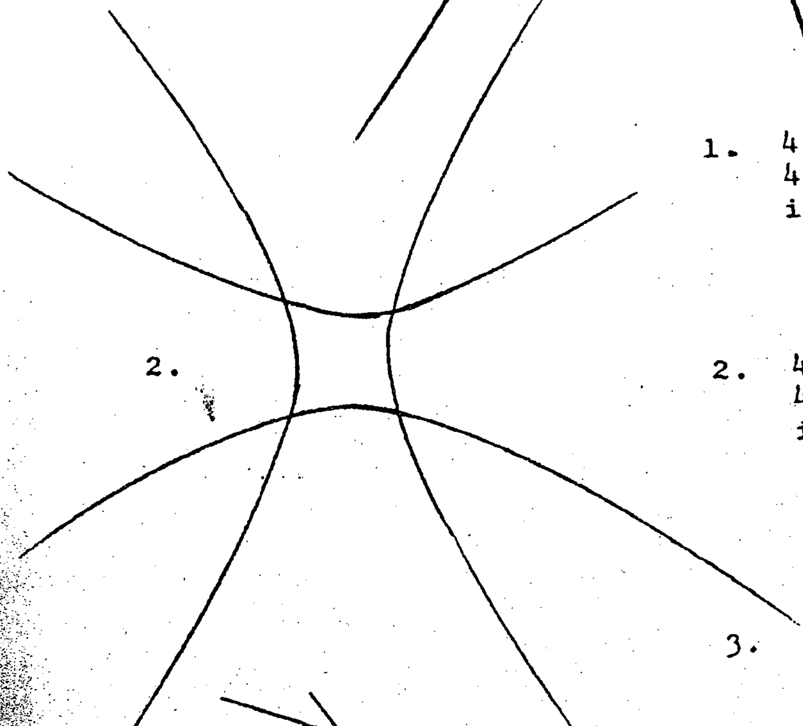
may have two real intersections ; then they have two common tangents.

They may also have:



1.

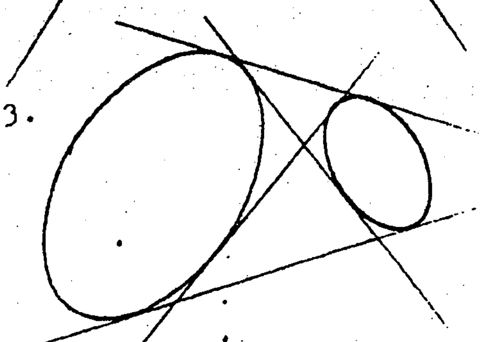
1. 4 real points and 4 real tangents in common



2.

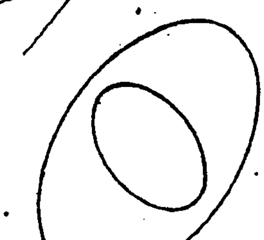
2. 4 real points and 4 imaginary tangents in common

3. 4 imaginary points and 4 real tangents in common



3.

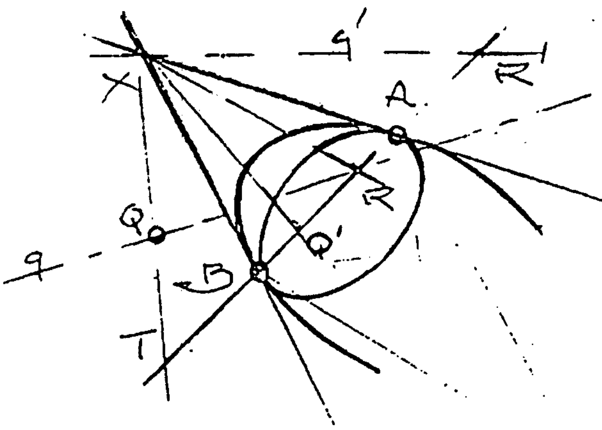
4. 4 imaginary points and 4 imaginary tangents in common.



4.

In case 4 the construction of the extra triangle fails. Take two lines p q, and construct the two eleven-point conics. These two conics intersect in the three vertices of the common self-polar triangle and also in the point conjugate to the intersection of pq with regard to both conics, which conj.pt. is always real. A further proof that one of the corners of the self-polar triangle is always real, as one real intersection of two conics must be paired with another, is thus given.

NICS with DOUBLE CONTACT



A and B are points of contact with fixed tangents----

Such a set of conics can be seen as a pencil of conics or as a range of conics.

R and R' are harm.conjugate with respect to A and B.

X R R'is the self-polar triangle.

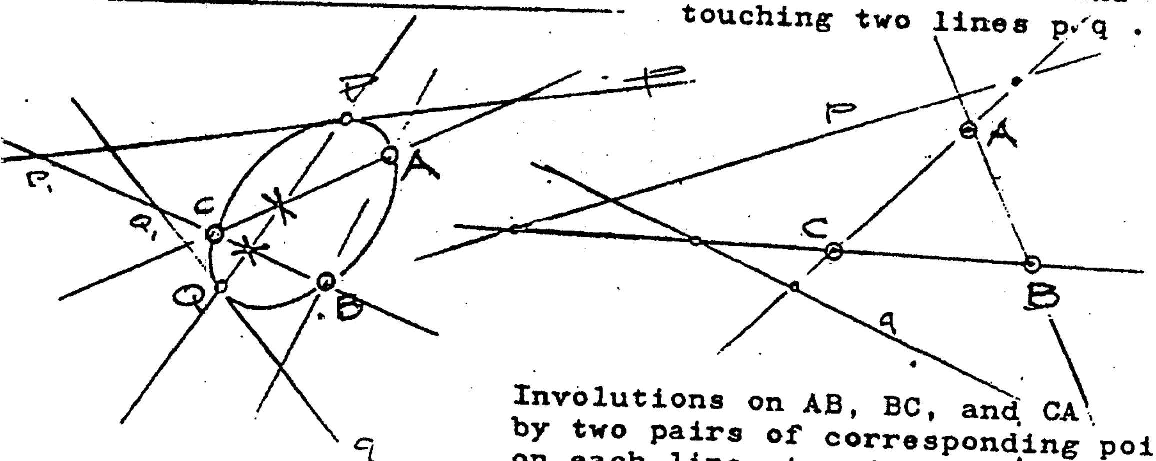
There is an infinity of polar triangles to the set.

This set of conics has both the properties of a pencil and of a range of conics.

|| The locus of poles of any line to the set is a straight line. The polars of a point pass through a point.

Consider any line q (RR' harm. to AB- XRR'is the self-polar triangle)q' is locus of poles of q w.r. to all conics .
XQ' is conjugate ray to XQ; Q'XT is the self-polar triangle.Q' is conjugate to Q w.r. to all conics.
The 11-point conic corresponding to q is 2 straight lines.
A-B locus of points conjugate to points on q, and
X-R' locus of poles.
In like manner: eleven-line conic corresponding to Q, becomes
- point X(which is the envelope of lines conjugate to lines through Q) and
point Q' (which is the envelope of polars of Q w.r. to the conics).

Construction of conics through three points A B C and touching two lines p q .



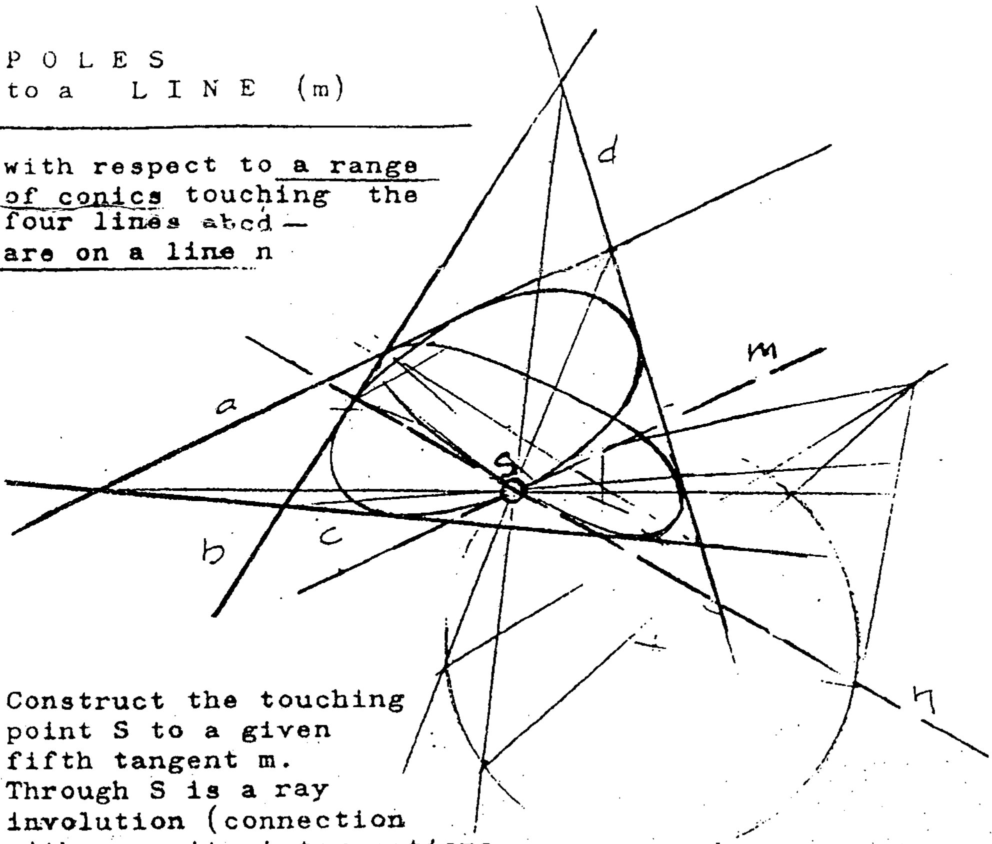
Involutions on AB, BC, and CA by two pairs of corresponding points on each line give double points on each of the three lines.

The connection P-Q must go through these double points.
4 connections are possible;this gives four solutions.

(w.r. = with respect)

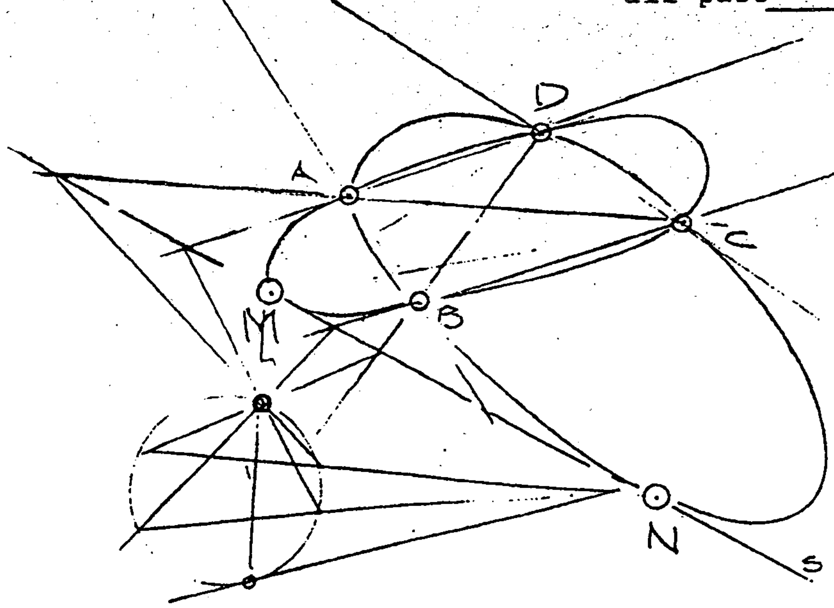
P O L E S
to a L I N E (m)

– with respect to a range of conics touching the four lines abcd – are on a line n



Construct the touching point S to a given fifth tangent m.
Through S is a ray involution (connection with opposite intersections of the tangents). The conjugate ray to m is the line n..
All the poles to the line m must be on this conjugate ray n.
As the lines m n are the double rays they are harm.to every pair of lines to opp. vertices. So they divide the diagonals harmoniously.

POLAR LINES to a POINT with respect to a pencil of conics all pass through a POINT.



Construct the tangent through a fifth point M.

Involution on s.

M is one double point; find the other one N which is the pencil of polar lines to the point M.

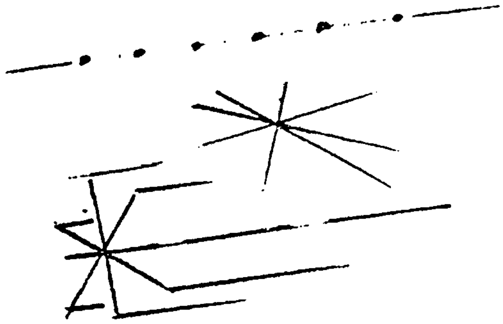
Projective Relations between Elementary Forms

One-dimensional elementary forms :

Range of points

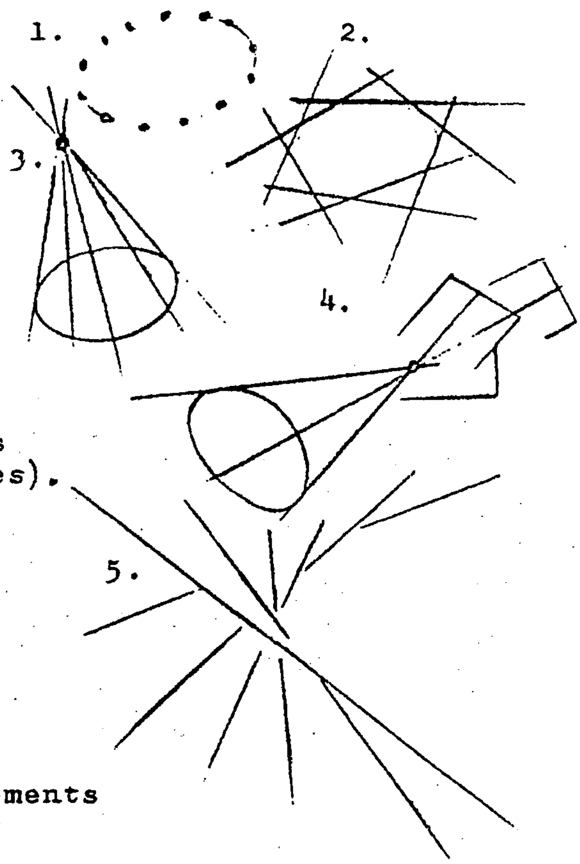
Flat Pencil
(pencil of lines)

Axial Pencil
(pencil of planes)



With these one-dimensional elementary forms five forms of the second order can be produced by projective relation:

1. Range of the second order
2. Pencil of the second order
3. Cone of lines (of the second order)
4. Pencil of planes of the second order - all the planes go through one point (cone of planes)
5. Regulus surface (produced by three skew lines or two projective skew ranges).



ig.2. Two projective ranges give a pencil of lines of 2nd order,

g.1. Two projective flat pencils in the same plane give a range of 2nd order,

---

Definitions of harmonious elements in forms of the second order:

4 harm. points of a range of 2nd order are projected by 4 harm. rays from a 5th point of the curve.

4 harm. rays of a pencil of 2nd order intersect a fifth ray in 4 harm. points.

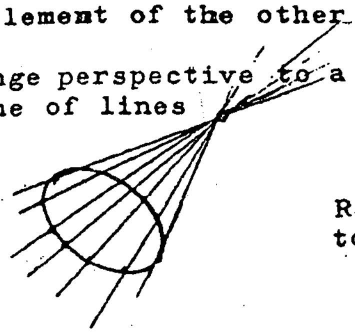
Cone of lines: from a fifth ray connect to the other rays to form 4 harm. planes.
Cone of planes: a fifth plane is intersected in 4 harm. lines.

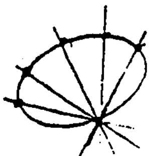
4 harm. rays in a regulus are intersected by every guideline in 4 harm. points and form with the guideline 4 harm. planes.

Two elementary forms are projective if 4 harm. elements of the one form correspond to 4 harm. elements of the other form (definition by von Staudt 1847).

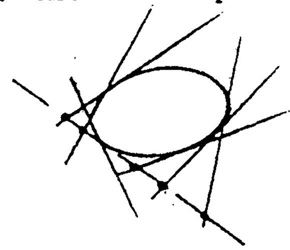
Two one-dimensional forms of different kind are perspective if every element of one form is part of the corresponding element of the other form.

Range perspective to a cone of lines



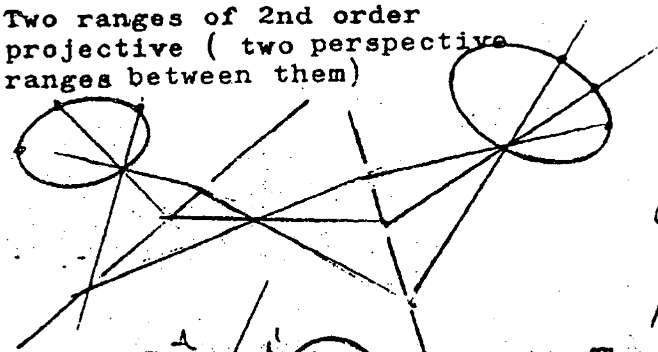


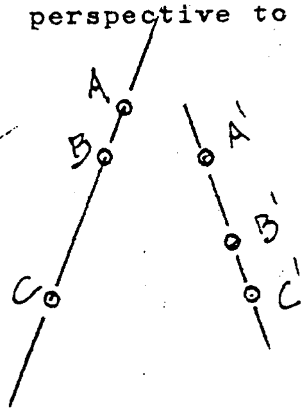
Range perspective to a pencil

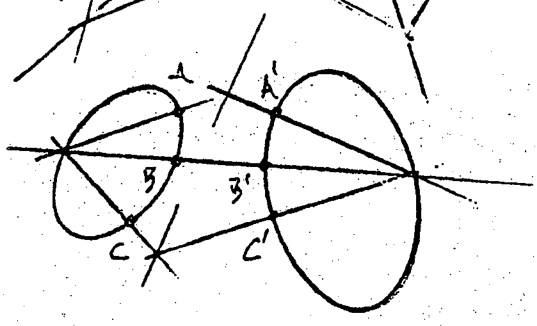


Pencil of 2nd order perspective to a range

Two ranges of 2nd order projective (two perspective ranges between them)

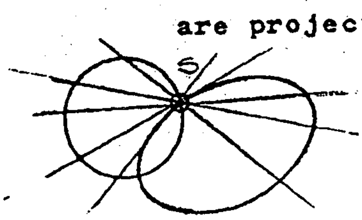


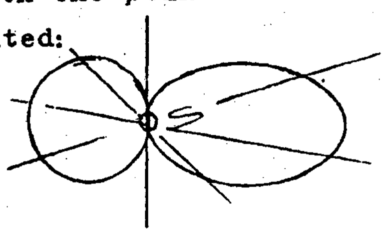




Three corresponding pairs of elements produce projectivity.

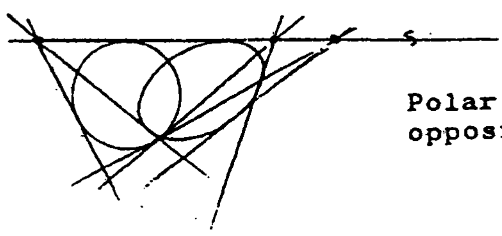
Two curves of 2nd order, with one point in common, are projectively related:



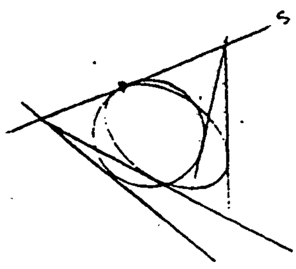


and perspective to the pencil in S.

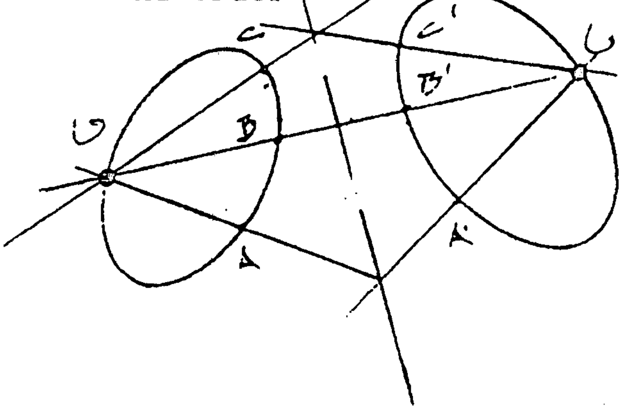
If touching in S, this is a self-corresponding point.



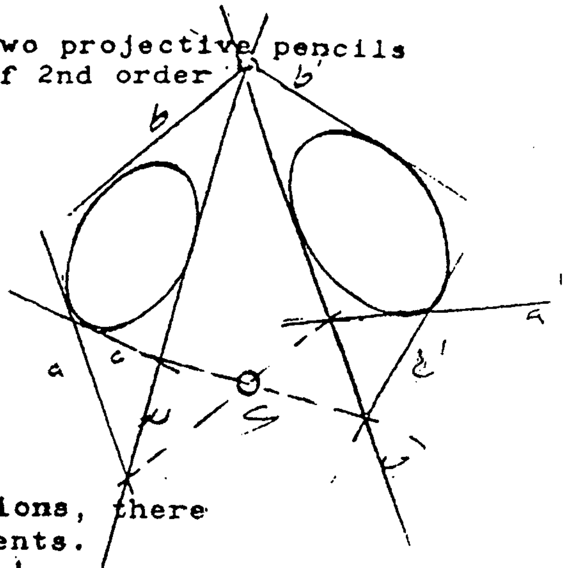
Polar opposites.



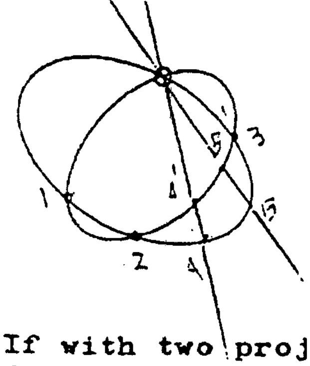
Two projective ranges of 2nd order

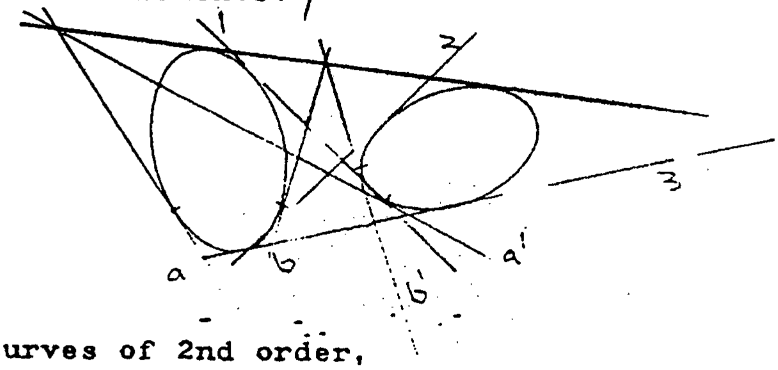


Two projective pencils of 2nd order



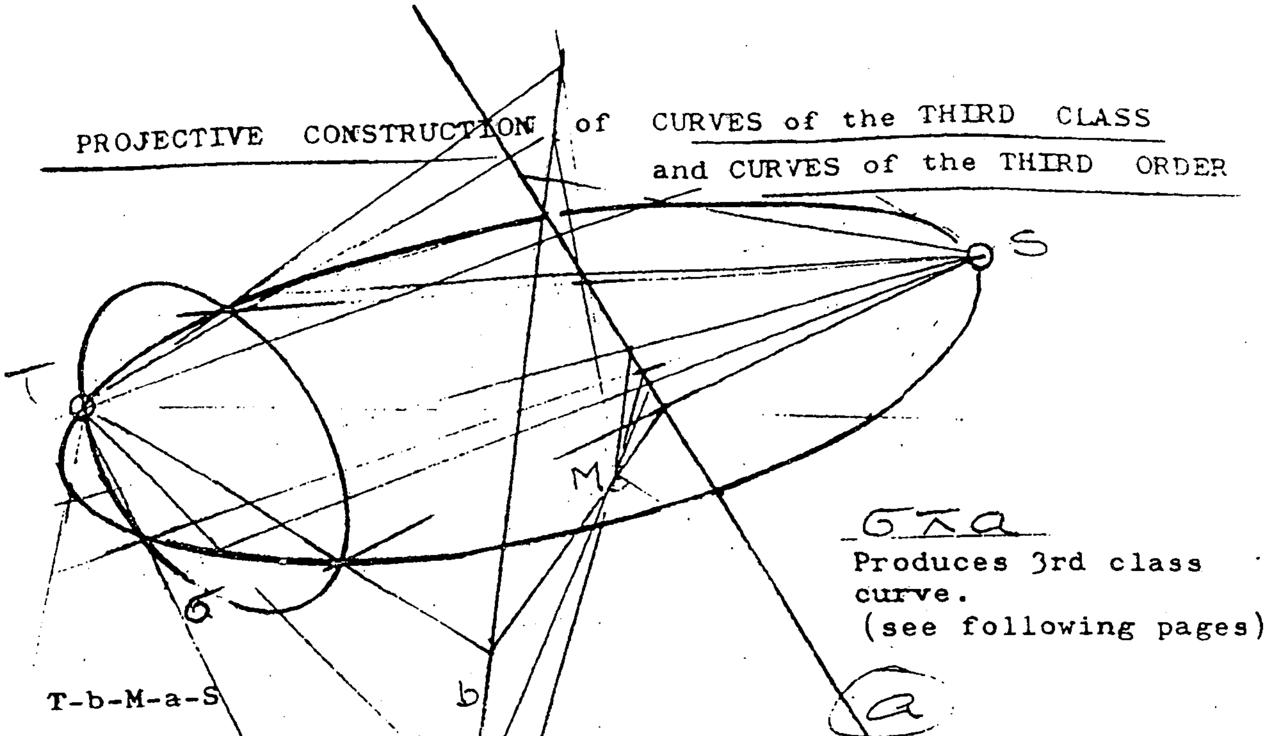
In the following projective relations, there are at maximum three common elements.





If with two projective curves of 2nd order, four common elements exist, then all corresponding elements are in common . The curves are identical.

PROJECTIVE CONSTRUCTION of CURVES of the THIRD CLASS and CURVES of the THIRD ORDER



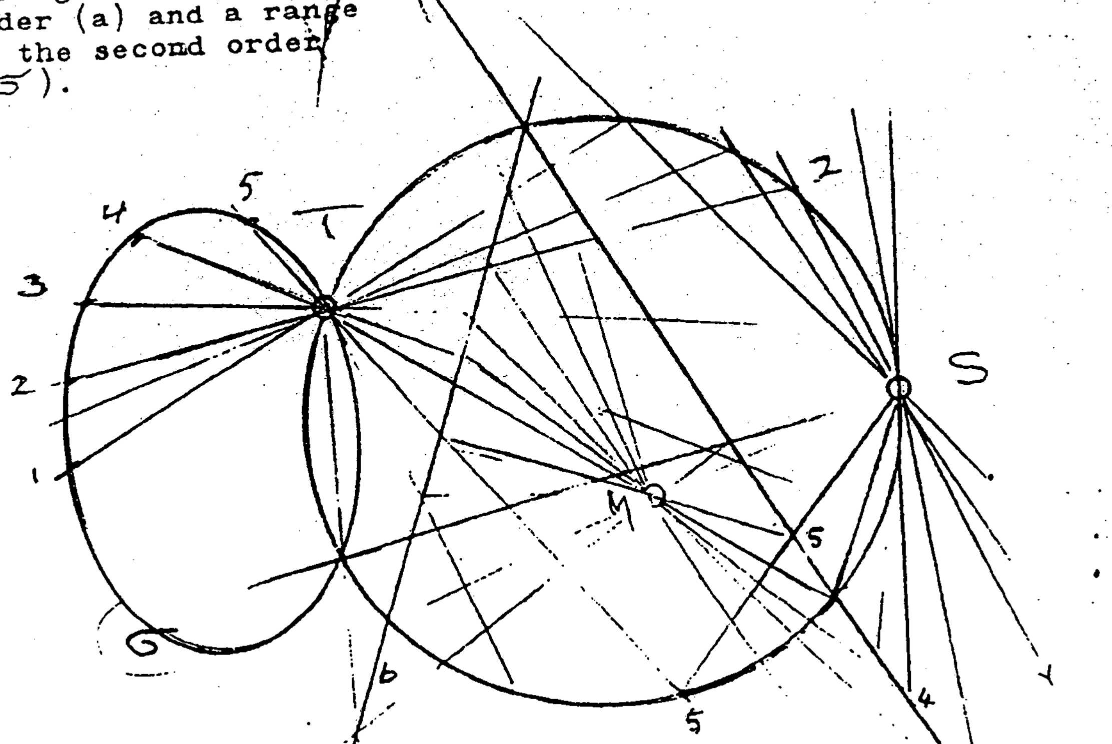
σ ⊼ a

Produces 3rd class curve.
(see following pages).

Pencil in S is projective to range on conic σ without being perspective to that range.
At least one ray and at most three rays of pencil S go through their corresponding points on conic σ besides the ray to point T.

This establishes a projectivity between a range of the first order (a) and a range of the second order (σ).

Pencil S is projective to pencil T.



The order of a curve is the maximum number of intersections with a line.

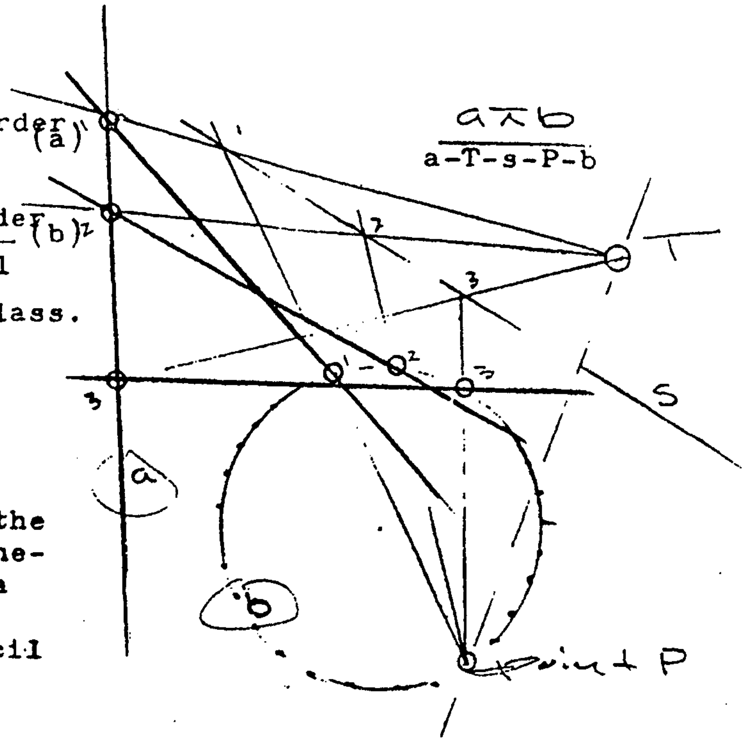
The class of a curve is the maximum number of tangents from a point.

A range of the first order (a) projective with a range of the second order (b) :

They produce in general a curve of the third class.

A point on the conic, an auxiliary line s and a pencil T produce the projectivity.

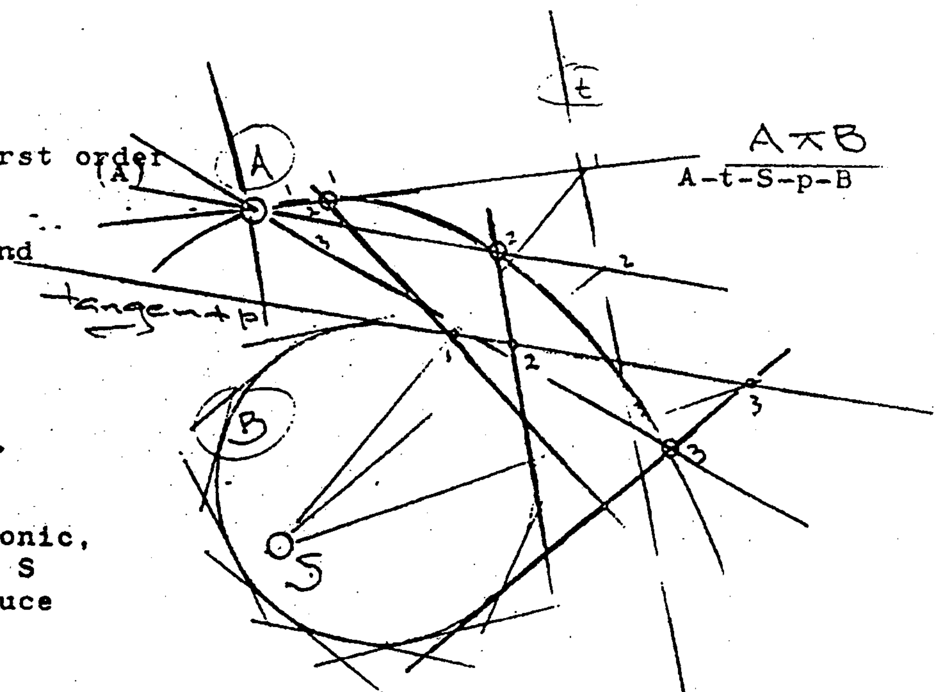
If the line a crosses the conic (outside the line-wise conic) a will be a double tangent to the resulting curve, a pencil of the third order.



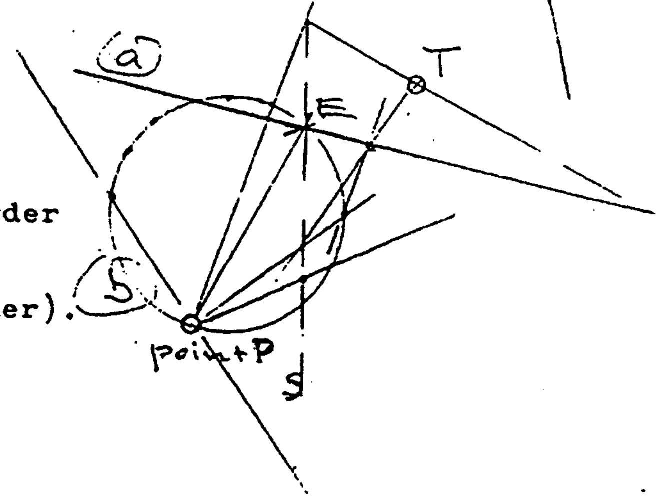
A pencil of the first order (A) projective with a pencil of the second order(B) :

They produce in general a curve of the third order.

A tangent on the conic, an auxiliary point S and a range t produce the projectivity.

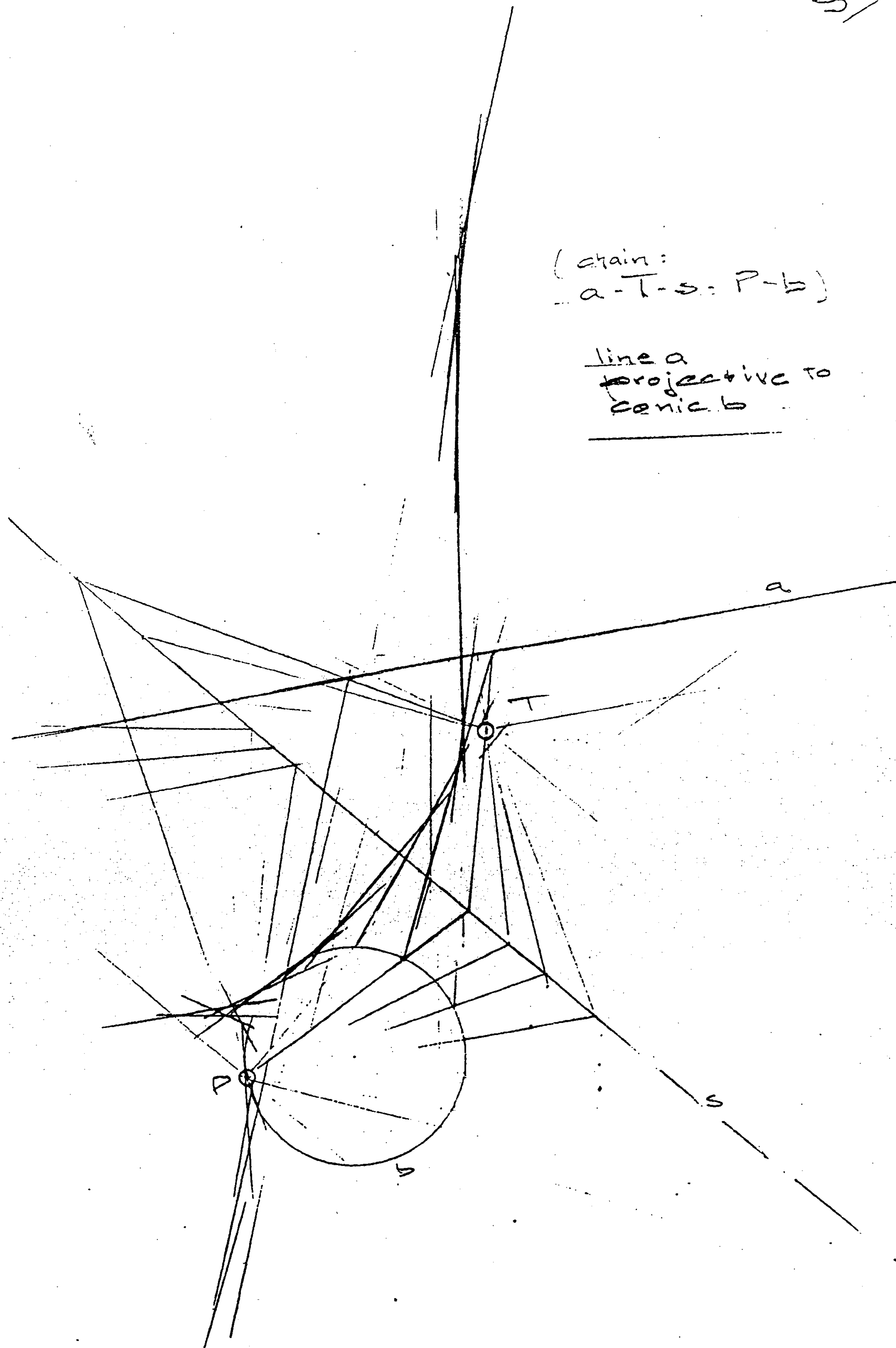


If the two projective ranges have a point E in common (2 corresponding points identical), the pencil of the third order degenerates to a pencil in E and a linewise conic (pencil of the second order).

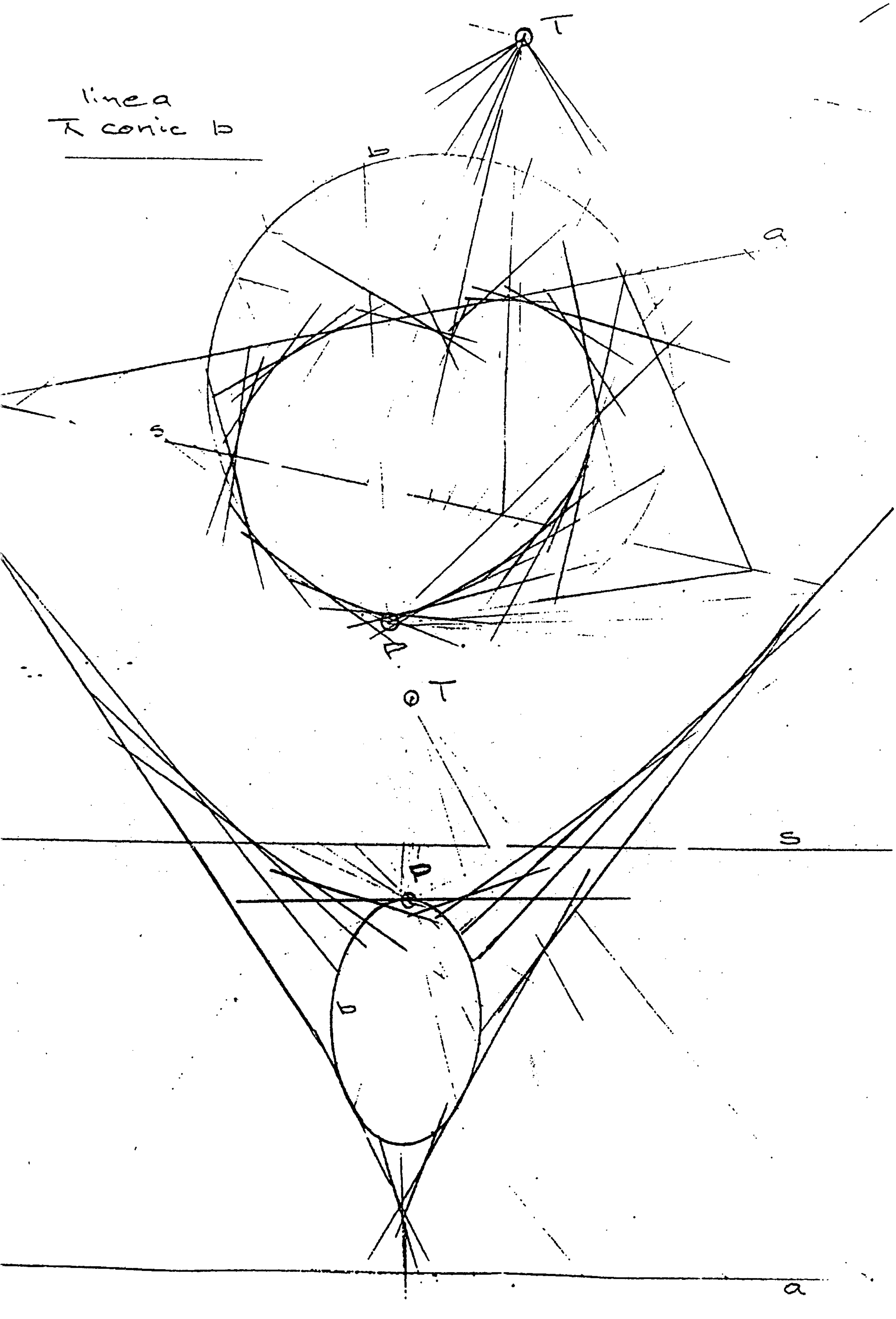


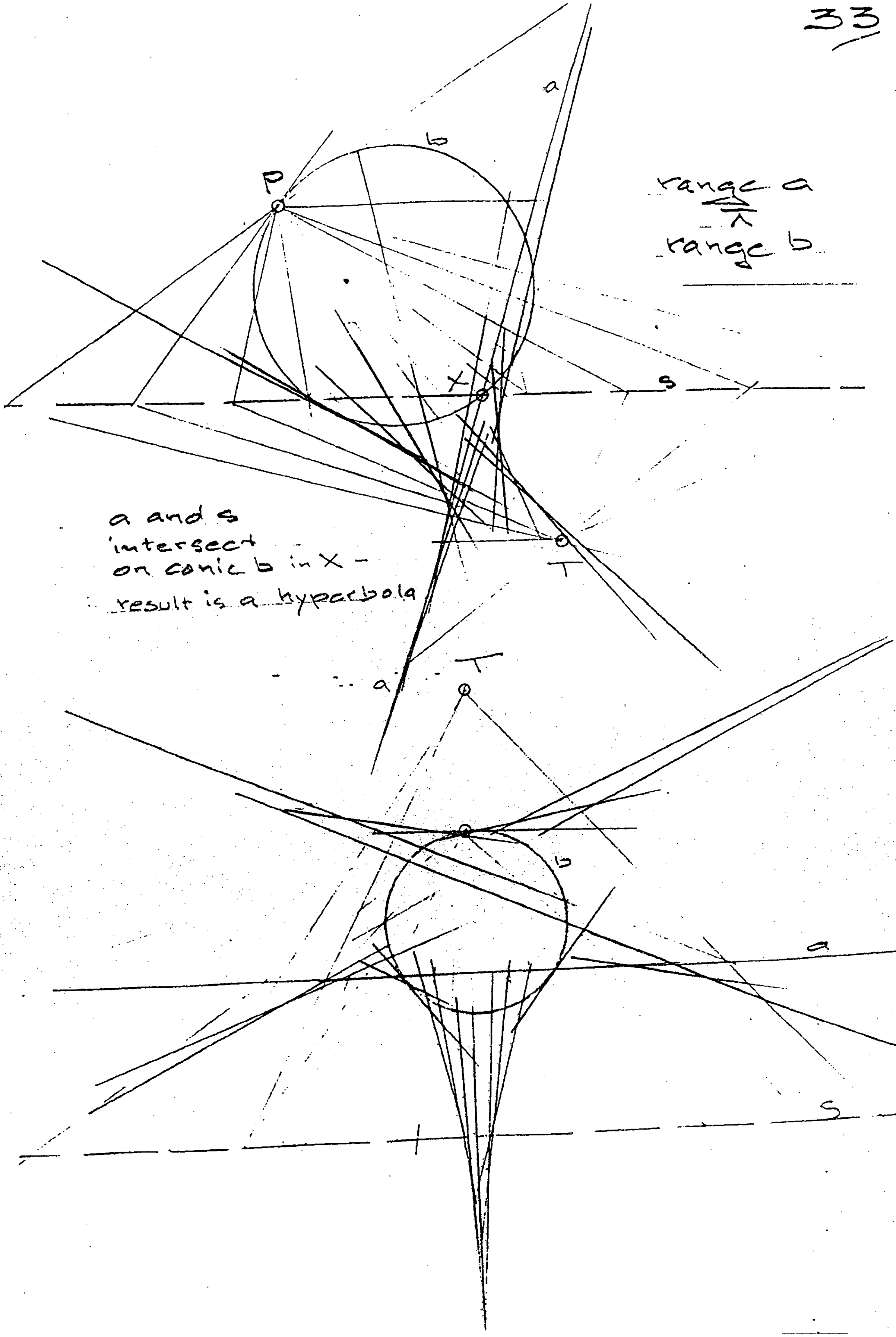
(chain:
a - T - s : P - b)

line a
projective to
conic b



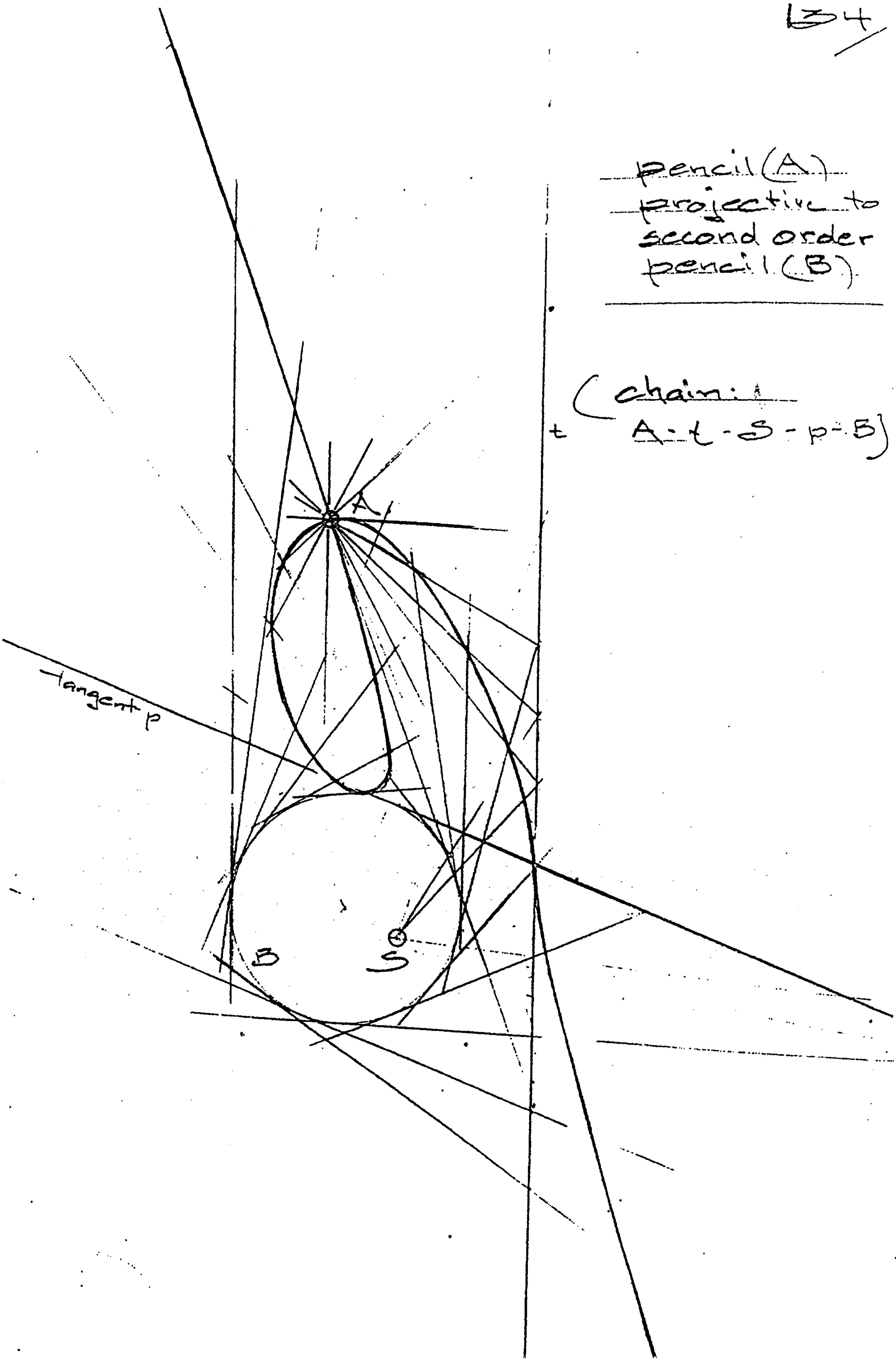
linea
⊼ conic b




range a
≅
range b
a
b
P
X
s
T
a and s
intersect
on conic b in X -
result is a hyperbola
a
T
b
a
s

pencil (A) projective to second order pencil (B)

(chain: A - t - S - p - B)



pencil(A) ⊼ pencil(B)